A Macro-Finance Model of Capital Reallocation and Misallocation *

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Abstract

I study how financial intermediaries' allocation of risk-bearing capacity shapes sectoral investment and capital allocation in a production general-equilibrium model. Leveraged intermediaries allocate wealth across equity claims on sectors that differ in productivity and exposure to aggregate shocks, subject to a limit on total portfolio risk. I find that sectoral reallocation feeds back into intermediary wealth dynamics and can slow the recovery of output growth after adverse shocks. Binding constraints reinforce this channel by redirecting allocation of financial risk and real capital away the unconstrained equilibrium. The investment distortions persist over a twenty-year simulation horizon, despite firms facing no financing constraints. Binding constraints also redistribute welfare: households incur losses, while intermediaries can gain. These welfare effects are less pronounced when capital across sectors is distributed more balancedly. The paper highlights intermediaries' role as *risk allocators*, showing how their allocation of risk-bearing capacity across assets shapes sectoral investment and the recovery of output growth.

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1 Introduction

The global financial crisis has underscored the importance of the financial system in the real economy. When leveraged intermediaries bear a large share of aggregate risk, adverse shocks to their balance sheets can impair risk bearing and amplify downturns (Brunnermeier and Sannikov, 2014). While we have a good understanding of how financial balance sheets determine the amount of capital investment and economic growth, this raises a question: how does intermediaries' allocation of risk-bearing capacity shape *where* capital is invested across productive sectors?

This question connects to a broader literature on how resource allocation shapes aggregate productivity and economic growth. Misallocation of capital and labor can generate substantial losses in aggregate productivity and output (Restuccia and Rogerson, 2008; Hsieh and Klenow, 2009). Barriers to reallocation can delay the transition to higher productivity (Buera and Shin, 2013) and also slow long-run economic growth (Acemoglu et al., 2018). Therefore I ask how intermediaries' risk-allocation decisions shape sectoral investment, and whether the resulting changes in capital composition feed back into intermediary balance sheets, asset prices, and economic growth.

I study these questions in a production general-equilibrium model linking financial risk allocation to sectoral investment. The paper makes two contributions. First, it proposes a risk-sharing channel of capital allocation. In the unconstrained benchmark, sectoral investment and allocation reflect both productivity and compensation for risk endogenously. Binding risk limits create different shadow costs across assets and distort sectoral valuations and investment relative to that benchmark. Second, it identifies a general-equilibrium feedback between capital composition and intermediaries' balance sheets. This feedback operates in both the unconstrained and constrained economies and can offset the contribution of higher risk premia to financial wealth rebuilding, slowing the recovery of output growth from distress. Together, they highlight intermediaries' role as *risk allocators*: their allocation of risk-bearing capacity shapes relative sectoral costs of capital, while the resulting capital composition affects their own wealth rebuilding.

To develop the mechanism formally, the model combines two productive sectors with different productivity and aggregate risk exposures and heterogeneous investors who differ in risk aversion and face portfolio risk constraints. The multi-sector structure allows financial frictions to affect the sectoral composition of investment as well as its aggregate level. Sectoral capital composition and intermediary capitalization jointly determine valuations, investment, and growth. The model abstracts from firm-level constraints, isolating the effects of constrained financial risk sharing on sectoral investment.

The production side builds on multi-sector models with costly capital adjustment, as in Eberly and Wang (2025) and Hansen et al. (2024). Capital is sector specific once installed. Changes in relative valuations affect sectoral investment, and adjustment costs make the resulting evolution of capital composition gradual. There is no direct transfer of installed capital between sectors. Both technologies produce the same consumption good. In the quantitative model, sector 1 is more productive and more exposed to aggregate risk than sector 2. Investors hold equity claims on the firms in both sectors. Orthogonal fundamental shocks isolate the comovement in equity returns that arises endogenously through valuations.

Investors have recursive preferences. Financial intermediaries (type A) are relatively risk tolerant, while households (type B) are more risk averse.[1] Both types trade the two equity claims and a risk-free bond. In equilibrium, intermediaries borrow from households and hold leveraged portfolios (He and Krishnamurthy, 2013; Brunnermeier and Sannikov, 2014). Multiple risky assets allow them to choose how to allocate their risk-bearing capacity across sectoral exposures. An institutional counterpart is the allocation of risk limits across trading desks: Bräuning et al. (2026) document spillovers between dealer-bank limits in different asset classes.

A VaR-type margin constraint places a bound on the total exposure of each investor to fundamental shocks (Kargar, 2021). Because equity return exposures depend on equilibrium valuations, the restriction on asset holdings varies with the state of the economy. It depends on covariances as well as individual asset volatilities. In the calibration, the limit occasionally binds for intermediaries while remaining slack for households, consistent with notion that leverage constraints are most relevant for financial intermediaries. Absent these constraints, investors freely allocate wealth across assets and share aggregate risks, and sectoral investment reflects equilibrium risk-adjusted costs of capital. Misallocation is measured relative to this unconstrained benchmark and arises from restrictions on investors' risk bearing.[2]

To clarify the misallocation in the paper, consider a textbook investment model with a common user cost of capital and no adjustment costs or compensation for aggregate risk. Let $MPK_{i,t+1}$ denote firm i's marginal product of capital, r the risk-free rate, and δ depreciation. Without firm-specific wedges, the investment condition is

$$\mathbb{E}_t[MPK_{i,t+1}] = r + \delta. \tag{1}$$

In this benchmark, dispersion in marginal products motivates the introduction of firm-level

[1] Intermediaries should be interpreted as representing leveraged institutions, such as commercial banks, investment banks, and hedge funds. Households correspond generically to institutions that provide short-term funding.

[2] In the quantitative model, the constraints bind in states with weak intermediary capitalization.

wedges,

$$\mathbb{E}_t[MPK_{i,t+1}] = r + \delta + \tau_{i,t}, \tag{2}$$

where $\tau_{i,t}$ denotes the wedge for firm i. Such wedges provide explanations for dispersion in marginal products, although substantial dispersion remains to be accounted for (David and Venkateswaran, 2019).

Allowing for compensation for aggregate risk changes the benchmark condition to

$$\mathbb{E}_t[MPK_{i,t+1}] = r + \delta + \beta_{i,t}\lambda_t, \tag{3}$$

where $\beta_{i,t}$ denotes exposure to aggregate risk and λ_t denotes the market price of the risk (David et al., 2022). Expected marginal products can differ across firms because their risk-adjusted costs of capital differ, even without firm-specific wedges.

The risk-premium term $\beta_{i,t}\lambda_t$ in equation (3) illustrates the channel studied in this paper. My model endogenizes allocation and pricing of risk: binding constraints on intermediaries' risk bearing shift sectoral risk premia, valuations, and investment relative to the unconstrained equilibrium. These investment distortions arise through financial risk compensation rather than firm-level wedges, while equilibrium asset-pricing conditions remain satisfied. The MPK conditions above provide motivating benchmarks. In my model, sectoral AK technologies imply fixed marginal products, while investment responds to equilibrium valuations through Tobin's Q. I measure distortions using deviations in sectoral Tobin's Q and investment rates from the unconstrained equilibrium at the same state.[3] Throughout the paper, misallocation denotes these constraint-induced investment deviations relative to the unconstrained benchmark.

The model's equilibrium dynamics are described by two endogenous state variables: x, the share of productive capital in sector 1, and w, intermediaries' share of aggregate wealth. Capital composition determines the mix of fundamental risks on the production side; wealth distribution affects how those risks are shared among investors. Both state variables influence portfolio choices, risk prices, and the valuations that guide investment. Their joint evolution links real capital accumulation to financial wealth dynamics.

Two opposing forces shape capital composition in the unconstrained economy. Consider an adverse shock to sector 1 when it accounts for a large share of capital and intermediaries are more exposed to its fundamental shock than households. The shock lowers both x and w. Holding composition fixed, lower intermediary wealth raises the price of sector 1's risk, reduces its valuation and investment, and increases investment in the safer sector. This response puts downward pressure on x. Holding wealth distribution fixed, a lower capital share

[3] Sectoral investment and capital growth are monotone functions of Tobin's Q in the model.

of sector 1 instead raises its investment relative to sector 2, generating a force that rebuilds its capital share. The balance-sheet response and the composition response therefore act in opposite directions on sectoral accumulation.

Capital composition also affects intermediary wealth rebuilding. As x falls in this region, intermediaries reduce exposure to sector 1's fundamental shock and increase exposure to the second shock. Risk prices and portfolio returns adjust with these choices. The numerical solution shows a lower expected rate of increase in intermediary wealth share at lower x, holding w fixed. Investment responses that change capital composition therefore also affect subsequent financial conditions. This return connection from productive composition to intermediary wealth is absent from a one-sector model, even though the usual connection between wealth, risk compensation, and aggregate investment remains.

Financial constraints add an investment distortion to these reallocation forces. If the adverse shock leaves intermediaries sufficiently weak, their total risk exposure reaches the limit and they cannot take the exposure they would choose without the restriction at prevailing prices. The constraint has a positive shadow cost, measured by its multiplier. For an individual asset, this cost is the multiplier times the asset's marginal contribution to portfolio risk. Consequently, a common limit creates different asset-specific shadow costs when claims contribute differently to the investor's total risk. In the region where sector 1 dominates, its equity carries a larger constraint wedge than sector 2's equity.

Relative to the unconstrained equilibrium at the same state, intermediaries bear less of sector 1's fundamental risk and households bear more. The equilibrium price of that risk rises. Sector 1's Tobin's Q and investment rate are lower, while sector 2's are higher. These are equilibrium responses to constrained risk sharing: firms continue to choose investment optimally given valuations and face no additional financing constraint. The asset-specific shadow costs identify the investor incentives behind the adjustment; they are distinct from the equilibrium changes in risk prices and premia.

Lower capital growth in sector 1 and higher growth in sector 2 reduce the drift of x relative to the unconstrained economy at the same state. This adds to the downward pressure on sector 1's capital share associated with weak intermediary wealth and works against the investment response that rebuilds the sector after its capital loss. In the low-wealth, high-x region, the constrained economy also has a lower drift of intermediary wealth share. Meanwhile, the investment-induced movement toward lower x leads toward states with weaker expected wealth rebuilding. The interaction of sectoral accumulation and financial wealth thus provides a channel through which investment distortions can affect subsequent financial conditions.

Forward simulations establish that the differences extend beyond the local drift compari-

son. Starting from a state with weak intermediary capitalization and a dominant risky sector, I simulate the constrained and unconstrained economies under the same future fundamental shocks, allowing both x and w to evolve. Mean intermediary wealth share recovers more slowly in the constrained economy. Sector 1's investment rate remains lower and sector 2's higher on average over the simulated horizon. These dynamic investment gaps reflect both the constraint's effects on policy functions and the different state paths that follow. They document persistent investment distortions and slower financial wealth rebuilding in this experiment.

Risk limits also redistribute welfare across investors. At the same initial physical capital, sectoral composition, and wealth share, households incur consumption-equivalent welfare losses, while intermediaries can gain at intermediate wealth shares. Thus, a restriction on intermediaries' portfolios need not reduce their equilibrium welfare: prices and investment opportunities adjust along with their feasible portfolios. Tighter limits raise the shadow cost of risk-bearing capacity and increase the sectoral investment distortions in the constrained region. They also produce larger household losses and can increase intermediary gains at intermediate wealth shares.

Alternative economies clarify the role of each modeling ingredient. In a one-sector economy with heterogeneous investors, intermediary capitalization and financial constraints affect risk premia and total amount of aggregate investment, but there is no capital-composition state and hence no capital reallocation. A multi-sector economy with a representative investor retains sectoral reallocation but leaves little room for the role of constrained risk-sharing. The full model combines both state variables and allows financial constraints to change the composition of investment as well as its aggregate level. Asymptotic expansions around these alternative economies provide tractable characterizations of these separate margins.

The joint dynamics of endogenous state variables x and w make the model strongly nonlinear. Since the equilibrium does not admit a closed-form solution, I solve it globally by casting the equilibrium conditions as a coupled system of second-order partial differential equations, extending the projection approach of Judd (1992, 2023) with neural-network approximations to prices and value functions over the full state space. The calibration disciplines the sectoral characteristics, investors' risk taking and where constraints bind. Relative to local perturbation or log-linear methods, this global solution captures captures changes in risk allocation across the state space and the transition between slack and binding constraints. Simulations then characterize the evolution of economy from various states.

These findings have broader policy implications and imply that capital requirements on intermediaries' total risk exposure can affect relative sectoral costs of capital unevenly and the redistribution of welfare across investors. A common risk limits links intermediaries' portfolio

choices across assets, allowing shocks to one sector to affect valuations and investment in another. The policy consequences of a tighter risk limit therefore depend on both investment responses and investor-specific welfare effects.

The remainder of the paper is organized as follows. Section 2 reviews the literature. Section 3 lays out the model, and Section 4 discusses the model's solution and the characterization of the equilibrium. Section 5 presents the quantitative analysis and discuss the results of the model. Finally, Section 6 concludes.

2 Related literature

This paper fits into several strands of literature. First, it contributes to the voluminous literature on financial frictions and macroeconomic dynamics, beginning with the seminal contributions of Bernanke and Gertler (1989), Bernanke et al. (1999), and Kiyotaki and Moore (1997). I build on the frameworks of He and Krishnamurthy (2013) and Brunnermeier and Sannikov (2014), which link financial-sector conditions to risk premia and asset prices. Recent contributions include Di Tella (2017), Maxted (2024), Krishnamurthy and Li (2025), and Gopalakrishna (2023). My model incorporates heterogeneous investors with recursive preferences and state-dependent portfolio constraints into a multi-sector production economy.

My paper studies how intermediaries allocate scarce risk-bearing capacity across sectoral equity claims and how this allocation shapes firms' investment. Intermediary wealth and the composition of productive capital jointly determine equilibrium portfolios, risk prices, and whether the VaR constraint binds. When the constraint binds, its effect on each equity claim depends on that claim's marginal contribution to portfolio risk. A common portfolio constraint can therefore affect relative valuations and investment across sectors. Productive capital remains sector specific: investors trade financial claims, while physical capital composition adjusts through investment. These investment responses change the risks investors must subsequently absorb, feeding back into equilibrium risk prices and intermediary wealth.

The paper also relates to empirical work showing that intermediary constraints operate across multiple asset markets. He et al. (2017) show that intermediary balance-sheet conditions price a broad cross-section of assets, and recent work provides micro-evidence that dealer constraints generate spillovers across asset markets (Bräuning et al., 2026). Related work on internal capital markets studies the allocation of scarce resources across competing projects (Stein, 1997). Banking studies show how funding and balance-sheet capacity are allocated across units and markets (Cetorelli and Goldberg, 2012a,b; Bräuning and Ivashina, 2020). My model connects intermediaries' allocation of risk-bearing capacity across sectoral equity claims to firms' investment through equilibrium risk prices and valuations.

There is a growing literature studying capital misallocation, early examples of which include Restuccia and Rogerson (2008) and Hsieh and Klenow (2009). The role of financial frictions on the firm side is studied by Buera et al. (2011), Midrigan and Xu (2014), and Moll (2014), among others. More recently, Ai et al. (2020) develop a contracting theory with intermediation to emphasize the amplifying role of exogenous financial shocks in capital misallocation. Dou et al. (2025) study the GE implications of misallocation through the valuation channel and endogenous growth.

David et al. (2022) consider aggregate risk in a partial equilibrium (PE) setting, and show that differences in exposure to aggregate risk can rationalize dispersion in marginal products of capital. van Binsbergen and Opp (2019) study how informational inefficiencies in asset prices distort real capital allocation and highlight the potential gains from correcting mispricing. My paper studies how constrained risk sharing under rational expectations affects capital allocation when risk compensation is determined by investors' portfolio choices and wealth distribution. Investors correctly understand the risks and investment opportunities, but binding portfolio constraints prevent the risk sharing attainable in the unconstrained economy. Improved information or valuation methods alone therefore cannot eliminate the resulting investment distortions. Endogenizing the SDF allows investment to affect equilibrium risk prices by changing the composition of productive capital and the risks investors must absorb. Intermediary wealth and portfolio allocations jointly determine whether the VaR constraint binds and its shadow cost, while each claim's marginal contribution to portfolio risk determines the asset-specific incidence of that cost. The resulting equilibrium valuations guide sectoral investment, feeding back into productive composition and future risk sharing.

On a technical note, this paper also contributes to the heterogeneous-agent asset pricing literature, for which Panageas (2020) provides an excellent survey. The seminal contributions of Dumas (1989, 1992), Wang (1996), Chan and Kogan (2002), Bhamra and Uppal (2009, 2014), Longstaff and Wang (2012), Gârleanu and Panageas (2015) study equilibrium in economies with two heterogeneous agents and different preference assumptions. Basak and Cuoco (1998), Gârleanu and Pedersen (2011), He and Krishnamurthy (2012, 2013), Chabakauri (2013), and Rytchkov (2014) examine the asset pricing implications of exogenous and endogenous portfolio constraints in economies populated by two heterogeneous agents with one or many assets where constrained agents have logarithmic or constant relative risk aversion (CRRA) preferences. More recent examples include Drechsler et al. (2018), Silva (2020), Kargar (2021), Schneider (2022), who study the role of heterogeneous agents with recursive preferences in various settings.

This paper follows by allowing for investors with recursive preferences and endogenous portfolio constraints, and generalizes to a multi-sector production economy with real frictions.

This insight draws on the pioneering works by Cochrane et al. (2008) and Martin (2013) study the rebalancing mechanism in general equilibrium and return comovement in economies with multiple Lucas 'trees'. These models assume infinite adjustment costs and do not allow capital movements. The other extreme of the spectrum, as in Cox et al. (1985), shows that capital adjusts immediately to ensure the balance of portfolios in the absence of adjustment costs. Eberly and Wang (2009, 2025) extend these models by endogenizing the size of 'trees' in a two-sector production economy, allowing for sluggish capital reallocation due to costly adjustment. Along with this line of literature, Duarte (2018) utilizes habit preference in a two-sector production setting to endogenize the risk aversion of a representative agent and shows this force prolongs economic downturns. Kozak (2022) examines bond-stock comovement in a similar economy with a representative agent, who has a recursive preference with an exogenously time-varying risk aversion. The multi-sector structure of my model builds on Eberly and Wang (2025) and is related to Duarte (2018) and Kozak (2022) by featuring asymmetric sectors with distinct cash-flow risks. I embed this structure in an economy where heterogeneous investors share risk subject to state-dependent portfolio constraints. Intermediary capitalization affects sectoral investment through the allocation and pricing of risk. Comparing constrained and unconstrained equilibria identifies the investment distortions associated with these constraints and their implications for capital-composition dynamics.

3 Model

In this section, I outline a two-sector general equilibrium model in continuous time. A representative firm in each sector uses capital to produce consumption goods, subject to investment adjustment costs. The real technologies employed by firms are subject to aggregate shocks. The economy is populated by a continuum of investors with recursive preferences, categorized into two classes: financial intermediaries and households. Financial intermediaries are less risk averse than households, although both types of investors have access to the financial market subject to portfolio constraints. The information structure of the economy is standard. Uncertainty is characterized in a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$, with a standard filtration $\mathscr{F}_t \subset \mathscr{F}, t \in [0, \infty)$.

In the remainder of this section, I first describe the economic setup and decision problems faced by firms and investors, and then define the competitive equilibrium.

3.1 Production

The productive side of the economy consists of two sectors, each using capital as the sole factor of production. Let K_n, I_n, and Y_n denote the capital stock, gross investment, and output, respectively, in sector n for $n \in \{1,2\}$. I first introduce the firms' technology, followed by their optimization problems.

3.1.1 Technology

The representative firm in each sector employs a constant returns-to-scale production technology, specifically an "AK" type.[4] The stock of capital $K_{n,t}$ produces output at rate

$$Y_{n,t} = A_n K_{n,t} \tag{4}$$

per unit of time, where A_n is a constant governing the productivity of capital.

Capital stock in each sector is exposed to exogenous aggregate Brownian shocks. The evolution of capital stock in sector n follows

$$\frac{dK_{n,t}}{K_{n,t}} = \phi_n(\iota_{n,t})dt + \boldsymbol{\sigma}_n' d\boldsymbol{B}_t, \tag{5}$$

where $\iota_{n,t} \equiv I_{n,t}/K_{n,t}$ is the investment-capital ratio in sector n at time t. $\boldsymbol{B} = \{(B_{1,t}, B_{2,t})' \in \mathbb{R}^2; \mathscr{F}_t, t \geq 0\}$ is a two-dimensional Brownian motion in the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, representing aggregate shocks to the capital stock. Such shocks may reflect stochastic depreciation, investment-specific, or capital quality shocks.[5] The vector $\boldsymbol{\sigma}_n$ is the exposure of capital in Sector n to the aggregate shocks. Later in calibration, I assume no correlation in fundamentals between two sectors. Thus, capital in sector n is effectively only exposed to $B_{n,t}$. This allows the model to highlight the correlation between asset returns that emerge endogenously, rather than the exogenously specified correlation in fundamentals.

Following Hayashi (1982), Jermann (1998) and others, the investment in each sector incurs adjustment cost that is homogeneous of degree one in investment and capital stock.[6] Function $\phi(\cdot)$ satisfies $\phi'(\cdot) > 0$ and $\phi''(\cdot) \leq 0$ on the admissible domain. I assume a quadratic

[4] See, for example, Kogan (2001, 2004), Eberly and Wang (2025), Pindyck and Wang (2013), and Brunnermeier and Sannikov (2014).

[5] See, for example, Brunnermeier and Sannikov (2014), and Maxted (2024). However, they do not focus on multi-sector setup, which is the focus of this paper. The vector notation is used to simplify the exposition.

[6] Homogeneous adjustment cost functions have been widely used in the q theory of investment literature. Hayashi (1982) showed that with homogeneous adjustment costs and perfect capital markets, marginal and average q are equal. Jermann (1998) integrated this type of adjustment costs into a real business cycle model.

adjustment cost function given by

$$\phi(\iota) = \iota - \delta - \frac{\theta}{2}(\iota)^2, \tag{6}$$

where θ is the adjustment cost parameter and δ is the depreciation rate of capital. There is no direct transfer of installed capital between sectors.

3.1.2 Optimization problems

Firms pay output net of investment as dividends to shareholders. The representative firm in each sector chooses the investment rate to maximize the expected discounted value of dividends, given the *stochastic discount factor (SDF)*. The value of the firm in sector n is given by

$$S_{n,t} = \sup_{\iota_{n,s}} \mathbb{E}_t \left[\int_t^\infty \frac{\Lambda_s}{\Lambda_t} (A_n K_{n,s} - \iota_{n,s} K_{n,s}) ds \right]$$
$$\text{s.t. capital evolution (5) and adjustment cost (6).} \tag{7}$$

where Λ_t denotes the SDF that is determined in equilibrium. The evolution of SDF can be written as

$$\frac{d\Lambda_t}{\Lambda_t} = -r_t dt - \boldsymbol{\eta}_t' d\boldsymbol{B}_t \tag{8}$$

where r_t is the risk-free rate and $\boldsymbol{\eta}_t$ is the vector of market price of risk, both determined in equilibrium.

3.2 Investors

The economy is populated by a continuum of investors in two classes, A and B. I start from the demographic structure and preference of the investors in the economy, followed by their budget and financial constraints. Then I define the optimization problems faced by investors.

3.2.1 Demographics and preferences

The aggregate population remains constant and is normalized to one. To obtain stationarity in the model, I follow Gârleanu and Panageas (2015) and consider a simple overlapping-generations (OLG) framework. All investors, regardless of age, preference, or wealth, face an exogenous and constant mortality rate $\kappa > 0$, and a fraction κ of new investors are born per

unit of time.[7] In aggregate, newborn investors inherit the wealth of their deceased parents on an equal per-capita basis. Of the newly born investors, a constant fraction $\bar{u}$ is of type A, while $1-\bar{u}$ is of type B. The OLG setup ensures the existence of a stationary distribution of wealth among investors.

All investors of two classes have recursive preferences with constant coefficient of relative risk aversion (RRA) and elasticity of intertemporal substitution (EIS).[8] Specifically, the preference is described by the stochastic differential utility (SDU) introduced by Duffie and Epstein (1992a,b). That is, an investor of type i for $i \in \{A, B\}$ has lifetime utility defined by

$$J_{i,t} = \mathbb{E}_t \left[\int_t^\infty f_i(C_{i,s}, J_{i,s}) ds \right], \tag{9}$$

where function $f(C,J)$ is known as the aggregator of the recursive utility for consumption C and the continuation value J. The aggregator function $f(C,J)$ is given by

$$f(C,J) = \frac{1-\gamma}{1-1/\psi} J \left[\left(\frac{C}{[(1-\gamma)J]^{1/(1-\gamma)}} \right)^{1-1/\psi} - (\hat{\rho}+\kappa) \right], \tag{10}$$

where the RRA is denoted as γ, and the EIS is denoted as ψ. For simplicity, assume $\gamma \neq 1$. The time-additive separable CRRA utility is a special case of the above recursive utility specification, when the RRA γ is equal to the reciprocal of the EIS ψ, i.e., $\gamma = 1/\psi$. Gârleanu and Panageas (2015) show that, due to mortality risk, the (effective) subjective discount rate is defined as $\rho \equiv \hat{\rho} + \kappa$, where $\hat{\rho}$ denotes the time impatience.

The two classes of investors have different RRA γ. I use A to denote financial intermediaries and B to denote households. In particular, households are more risk averse than financial intermediaries, i.e., $\gamma_A < \gamma_B$. It follows that in equilibrium financial intermediaries, including banks, hedge funds, and broker-dealers, etc., raise funds from households via short-term debt financing and use these funds to make levered investments. To focus on the risk aversion heterogeneity, I assume the same EIS across all investors.

3.2.2 Budget and financial constraints

All investors in the economy can trade claims on dividends from the two sectors and a risk-free bond in zero net supply, subject to portfolio constraints. Investors dynamically choose their consumption and portfolio weights. Let s denote the birth date of an investor and $W_{i,t}(s)$ denote the wealth of an investor of type i born in date s at time t. Since investors of the same type face the same optimization problems (defined below), the decisions do not depend on

[7] The total population size is thus $\int_{-\infty}^{t} \kappa e^{-\kappa(t-s)} ds = 1$.

[8] See, for example, Kreps and Porteus (1978), Epstein and Zin (1989), and Weil (1989).

the age of each investor. Thus, the notation can be simplified by dropping s in the wealth dynamics. Let $\boldsymbol{\omega}_{i,t}$ be the vector of her portfolio weights in two risky assets. Then the financial wealth of an investor of type i evolves according to the dynamic budget constraint

$$\frac{dW_{i,t}}{W_{i,t}} = [\boldsymbol{\omega}_{i,t}'\boldsymbol{\lambda}_{R,t} + r_t - c_{i,t}]dt + \boldsymbol{\omega}_{i,t}'\boldsymbol{\Sigma}_{R,t}d\boldsymbol{B}_t, \tag{11}$$

where $c_{i,t} \equiv C_{i,t}/W_{i,t}$ is investor's consumption-wealth ratio. r_t is the risk-free rate, $\boldsymbol{\lambda}_{R,t}$ is the vector of expected excess return of risky assets, and $\boldsymbol{\Sigma}_{R,t}$ is the matrix of loadings on the aggregate shocks of asset returns (all defined in equilibrium in Section 4).

The leverage of an investor is defined as the ratio of asset over equity. In equilibrium, intermediaries lever up their balance sheets by borrowing from households. That is, when the sum of portfolio weights in two risky assets exceed one, the intermediaries lever up by issuing instantaneous risk-free debt to households. I assume investors face occasionally binding margin constraints, which depend inversely on the volatilities of returns of risky assets. Because return volatilities also depend on the state of the economy, margin requirements are state dependent as well.

I consider a change of variables that transforms the characterization to simplify the exposition. Define the risk exposure of an investor of type i as $\boldsymbol{\sigma}_{i,t}' \equiv \boldsymbol{\omega}_{i,t}'\boldsymbol{\Sigma}_{R,t}$. Given the absence of arbitrage, the vector of expected excess returns of risky assets is given by $\boldsymbol{\lambda}_{R,t} = \boldsymbol{\Sigma}_{R,t}\boldsymbol{\eta}_t$. The dynamic budget constraint can be rewritten as

$$\frac{dW_{i,t}}{W_{i,t}} = [\boldsymbol{\sigma}_{i,t}'\boldsymbol{\eta}_t + r_t - c_{i,t}]dt + \boldsymbol{\sigma}_{i,t}'d\boldsymbol{B}_t. \tag{12}$$

Following the change of variables, I assume financial constraint in terms of the risk exposure as

$$\|\boldsymbol{\sigma}_{i,t}\| \leq \zeta, \tag{13}$$

where $\|\cdot\|$ denotes the norm operator and measures the total risk exposure of investors. ζ denotes the tightness of the constraint on the total risk exposure.[9] Therefore, the investors effectively face a VaR rule, as in Danielsson et al. (2012), Adrian and Shin (2014), and Kargar (2021).[10] I extend the framework to multiple assets and multiple shocks, which allows for a general form of VaR constraints. As will be shown, this friction in the financial market is the

[9] This is equivalent to the equations (6) and (7) of Kargar (2021), in which the portfolio weight is bounded by $\frac{1}{\alpha\sigma_{R,t}}$ and $\sigma_{R,t}$ denotes the volatility of the risky asset return. The risk exposure is thus bounded by a constant $\frac{1}{\alpha}$. Note that investors will not short the risky asset in equilibrium if there is one single asset.

[10] VaR constraints, common for financial institutions in the Basel II and Basel III regulatory frameworks, aim at limiting downside risk and maintaining an equity cushion large enough so that the default probability is kept below some benchmark level.

key driver of allocative efficiency on the real side of the economy.

3.2.3 Optimization problems

Investors within each type face identical optimization problems, as mentioned above. They start with initial wealth $W_{i,0} > 0$, decides on consumption as a fraction of wealth $c_{i,t}$, and risk exposure $\boldsymbol{\sigma}_{i,t}$ to maximize lifetime utility. The optimization problem of an investor of type i is given by

$$\begin{aligned} V_{i,t} &= \max_{\{c_{i,s}\geq 0, \boldsymbol{\sigma}_{i,s}\}} J_{i,t} \\ &\text{s.t. dynamic budget constraint (12) and financial constraint (13),} \end{aligned} \tag{14}$$

and a solvency constraint $W_{i,t} \geq 0$.

3.3 Equilibrium

The following defines the competitive equilibrium.

Definition 1 *A competitive equilibrium is a set of adapted stochastic processes for interest rate r_t, market price of risk $\boldsymbol{\eta}_t$, stochastic discount factor Λ_t; firms' output $Y_{n,t}$, capital $K_{n,t}$, investment $I_{n,t}$, and value $S_{n,t}$ for $n \in \{1,2\}$; investors' wealth $W_{i,t}$, consumption $C_{i,t}$, risk exposure $\boldsymbol{\sigma}_{i,t}$ for $i \in \{A,B\}$ such that*

i. firms solve an optimal investment problem (3.1.2);

ii. investors maximize utility by choosing consumption and risk exposure (3.2.3);

iii. markets clear:

$$\begin{aligned} \bar{u}C_{A,t} + (1-\bar{u})C_{B,t} &= Y_{1,t} + Y_{2,t} - I_{1,t} - I_{2,t} \quad \textit{(goods market)}, \\ \bar{u}W_{A,t} + (1-\bar{u})W_{B,t} &= S_{1,t} + S_{2,t} \quad \textit{(bond market)}, \\ \begin{pmatrix} \boldsymbol{\sigma}_{A,t} & \boldsymbol{\sigma}_{B,t} \end{pmatrix} \begin{pmatrix} \bar{u}W_{A,t} \\ (1-\bar{u})W_{B,t} \end{pmatrix} &= \begin{pmatrix} \boldsymbol{\sigma}_{R_1,t} & \boldsymbol{\sigma}_{R_2,t} \end{pmatrix} \begin{pmatrix} S_{1,t} \\ S_{2,t} \end{pmatrix} \quad \textit{(equity markets)}. \end{aligned} \tag{15}$$

The third condition equates investors' aggregate risk exposures with those supplied by outstanding equity. Using $\boldsymbol{\sigma}_{i,t} = \boldsymbol{\Sigma}'_{R,t}\boldsymbol{\omega}_{i,t}$, it is equivalent to clearing both equity markets when $\boldsymbol{\Sigma}_{R,t}$ is nonsingular.

4 Solution

The equilibrium can be characterized in a recursive formulation, where all equilibrium objects are functions of aggregate state variables. In this section, I first define endogenous state variables and derive their dynamics. Subsequently, I derive the Hamilton-Jacobi-Bellman (HJB) equations, characterize the solutions to the optimization problems of firms and investors, and provide intuition regarding the optimal policy functions. Then I define a recursive Markov equilibrium and discuss risk-taking decisions and asset pricing in the unconstrained economy. I next explain how equilibrium risk pricing guides capital allocation in both constrained and unconstrained economies. Lastly, I use asymptotic expansion methods to characterize alternative economies to isolate key mechanisms. The full model with occasionally-binding constraints, which is solved numerically, will be discussed in Section 5.

4.1 State variables

Due to the homogeneity of Epstein-Zin preferences and the capital accumulation processes, all variables can be rescaled by the total wealth or total capital. Therefore, it suffices to track only two aggregate state variables: the share of the aggregate wealth that belongs to the investors of type A and the share of total capital that belongs to the firm in sector 1. Thus, the equilibrium conditions can be expressed as functions of these two endogenous state variables:

$$w_t = \frac{\bar{u} W_{A,t}}{\bar{u} W_{A,t} + (1-\bar{u}) W_{B,t}}, \qquad x_t = \frac{K_{1,t}}{K_{1,t} + K_{2,t}}. \tag{16}$$

This paper highlights the joint dynamics of w and x as the main drivers of the equilibrium, and the interaction between the two state variables as a key channel through which financial frictions affect the efficiency of capital allocation. The state variable w is one of the key variables in the macro-finance models (e.g., He and Krishnamurthy (2013) and Brunnermeier and Sannikov (2014)).[11] In these models, shocks to the aggregate capital stock effectively generate fluctuations in the wealth share of intermediaries, which in turn drive the dynamics of market price of risk and asset prices (as shown in Appendix C). Thus, the existing macro-finance models do not explicitly model the capital reallocation across sectors, which is captured by the state variable x in this model. The variable x has been studied in the literature on production-based models with multiple sectors (e.g., Eberly and Wang (2009), Duarte (2018) and Kozak (2022)). In these models, the representative investor allocates capital across sectors in response to shocks to the sectoral capital stock (as shown in Appendix C).

[11] Note that the combination of heterogeneity and recursive preferences gives rise to the wealth share as a state variable even under perfect risk sharing.

The following proposition gives the dynamics of state variable w_t, the wealth share of investors who belong to type A.

Proposition 1

$$dw_t = \kappa(\bar{u} - w_t)dt + w_t(1 - w_t)[\mu_{w,t}dt + \boldsymbol{\sigma}'_{w,t}d\boldsymbol{B}_t], \tag{17}$$

where $\mu_{w,t}$ and $\boldsymbol{\sigma}_{w,t}$ are given by

$$\mu_{w,t} = (\boldsymbol{\sigma}'_{A,t} - \boldsymbol{\sigma}'_{B,t})\boldsymbol{\eta}_t + c_{B,t} - c_{A,t} - (\boldsymbol{\sigma}'_{A,t} - \boldsymbol{\sigma}'_{B,t})\boldsymbol{\Sigma}'_{R,t}\begin{pmatrix} z_t \\ 1 - z_t \end{pmatrix},$$

$$\boldsymbol{\sigma}_{w,t} = \boldsymbol{\sigma}_{A,t} - \boldsymbol{\sigma}_{B,t}.$$

Proof. See Appendix A. ■

Notice that the law of motion of the intermediaries' wealth share w_t has an exogenous component in the drift function due to the demographic structure assumed earlier. This term ensures a non-degenerate long-run distribution of w, i.e., $w \in (0,1)$. The endogenous component of (17) is driven by the consumption and risk exposure choices of investors of both types (in brackets). Term $\mu_{w,t}$ depends on the consumption-wealth ratio of households relative to that of intermediaries, and on the differential aggregate risk exposure between intermediaries and households. Term $\boldsymbol{\sigma}_{w,t}$ only depends on this latter force. As will be shown in (31), the wealth share dynamics depend on asset prices, which themselves depend on wealth share dynamics. The two-way dependence amplifies the return volatility in the two-sector model, extending the result in Brunnermeier and Sannikov (2014).

The purpose of modeling two productive sectors is a theoretical device to showcase a continuum of capital with various degrees of riskiness across all firms in the economy. For exposition, I assume sector 1 is riskier and more productive, while sector 2 is safer but less productive.[12] Thus, sector 1 and sector 2 can be interpreted as the growth sector and the value sector, respectively. Additionally, the fundamental correlation between the capital stocks of the two sectors is assumed to be zero. This is not meant to capture empirical correlations, but allows to focus on capital reallocation between two distinct sectors and on the endogenous correlation in asset returns.

Assumption 1 *The risk exposure of sector 1, $\boldsymbol{\sigma}_1 = (\sigma_1, 0)'$, the risk exposure of sector 2, $\boldsymbol{\sigma}_2 = (0, \sigma_2)'$, and the overall risk exposure of sector 1 is also higher than that of sector 2, $\sigma_1 > \sigma_2$. The productivity of sector 1 is higher than that of sector 2, $A_1 > A_2$.*

The following proposition gives the dynamics of state variable x_t, the share of capital that belongs to the firm in sector 1.

[12] The assumption about productive sectors is a general extension on Duarte (2018) and Kozak (2022).

Proposition 2

$$dx_t = \mu_{x,t}dt + x_t(1-x_t)\sigma_1 dB_{1,t} - x_t(1-x_t)\sigma_2 dB_{2,t}, \tag{18}$$

where the drift of x_t *is given by*

$$\mu_{x,t} = x_t(1-x_t)\left[g_{1,t} - g_{2,t} + (1-x_t)\sigma_2^2 - x_t\sigma_1^2\right].$$

$g_{n,t}$ *denotes the growth rate of capital in sector* n *at time* t*.*

Proof. See Appendix A. ■

Note that through endogenous capital reallocation, $x_t \in [0,1]$. The endogenous drift term of (18) depends on the relative growth rates of capital in two sectors, adjusted by the difference in sectoral exposure to shocks. The volatility term is a quadratic function in x which attains its highest value at $x = 1/2$. Meanwhile, $x = 0$ and $x = 1$ correspond to the states where all capital is held by the firm in sector 2 and sector 1, respectively.[13] Using the definition of x, the expected growth rate of aggregate capital can be written as

$$g_t = x_t g_{1,t} + (1-x_t)g_{2,t}. \tag{19}$$

Similarly, let ι_t denote the aggregate investment rate, i.e., $\iota_t \equiv (I_{1,t} + I_{2,t})/(K_{1,t} + K_{2,t})$. The aggregate investment rate can be written as

$$\iota_t = x_t \iota_{1,t} + (1-x_t)\iota_{2,t}. \tag{20}$$

The aggregate quantities g_t and ι_t, which are functions of the state variables (x, w), are particularly important for understanding the model's implications for the business cycle.

It is also useful throughout to define another variable, $z_t \equiv S_{1,t}/(S_{1,t} + S_{2,t})$, the ratio of the firm value in sector 1 to the total value of firms in two sectors. Variable z_t, not a state variable *per se*, captures the weight of sector 1 in the market portfolio, and is a function of x_t and the relative valuation of firms in two sectors, as shown later.

4.2 Hamilton-Jacobi-Bellman equations

The recursive formulation of the equilibrium is characterized by the value function of firms and investors. This section derives the value functions that satisfy the HJB equations.

[13] These two states are absorbing states, as the sector cannot be rebuilt once it vanishes, because the loadings on the drift and diffusion terms of the capital stock evolution processes are zero.

4.2.1 Firms' problems

The firm's problem in each sector $n \in \{1,2\}$ is to maximize the expected discounted value of dividends $D_{n,t} \equiv A_n K_{n,t} - \iota_{n,t} K_{n,t}$, which can be written recursively

$$0 = \max_{\iota_{n,t}} \left\{ D_{n,t}\, dt + \mathbb{E}_t \left[\frac{d(S_{n,t}\Lambda_t)}{\Lambda_t} \right] \right\}. \tag{21}$$

The homogeneity property implies that marginal q equals Tobin's average q, following Hayashi (1982). The following proposition gives the firms' value functions.

Proposition 3 *The value of firm in sector n for $n \in \{1,2\}$ has the form*

$$S_{n,t}(K_{n,t}, x_t, w_t) = K_{n,t}\, q_{n,t}(x_t, w_t), \tag{22}$$

where $q_{n,t}$ captures future investment opportunities of the firm in sector n. The dynamics of $q_{n,t}$ is given by

$$\frac{dq_{n,t}}{q_{n,t}} = \mu_{q_n,t}\, dt + \boldsymbol{\sigma}'_{q_n,t}\, d\boldsymbol{B}_t, \tag{23}$$

where $(\mu_{q_n,t}, \boldsymbol{\sigma}_{q_n,t})$ is determined in equilibrium. The expressions for $\mu_{q_n,t}$ and $\boldsymbol{\sigma}_{q_n,t}$ are given in the Appendix A.

The HJB equation for the firms' problem is given by the following partial differential equation (PDE):

$$0 = \max_{\iota_{n,t}} \left\{ \frac{A_n - \iota_{n,t}}{q_{n,t}} + g_{n,t} + \mu_{q_n,t} + \boldsymbol{\sigma}'_{q_n,t}\boldsymbol{\sigma}_n - r_t - \boldsymbol{\eta}'_t(\boldsymbol{\sigma}_{q_n,t} + \boldsymbol{\sigma}_n) \right\}, \tag{24}$$

where $g_{n,t}$ denotes the expected growth rate of the capital stock in sector n.

Proof. See Appendix A. ■

The first-order condition (FOC) gives the optimal investment rate as follows

$$g'(\iota_{n,t}) = \frac{1}{q_{n,t}}. \tag{25}$$

Consistent with Q-theory of investment, the optimal investment rate is an increasing function of Tobin's Q, which summarizes the marginal benefit of increasing capital stock. In equilibrium, the representative firm in each sector optimally chooses to equate marginal cost to marginal benefit.

It is straightforward to derive derive aggregate Tobin's Q, which is given by

$$q_t = x_t q_{1,t} + (1 - x_t) q_{2,t}. \tag{26}$$

Note that the aforementioned variable z_t, which captures the weight of sector 1 firm value in the market portfolio, can be written as

$$z_t = \left(1 + \frac{q_{2,t}}{q_{1,t}} \frac{1 - x_t}{x_t}\right)^{-1}. \tag{27}$$

The return of holding claims on the firm is given by the dividend yield plus capital gains, as follows

$$dR_{n,t} = \frac{A_n - \iota_{n,t}}{q_{n,t}} dt + \frac{dS_{n,t}}{S_{n,t}}, \tag{28}$$

$$\equiv \mu_{R_n,t} dt + \boldsymbol{\sigma}'_{R_n,t} d\boldsymbol{B}_t, \tag{29}$$

where

$$\mu_{R_n,t} = \frac{A_n - \iota_{n,t}}{q_{n,t}} + g_{n,t} + \mu_{q_n,t} + \boldsymbol{\sigma}'_{q_n,t} \boldsymbol{\sigma}_n, \tag{30}$$

$$\boldsymbol{\sigma}_{R_n,t} = \boldsymbol{\sigma}_{q_n,t} + \boldsymbol{\sigma}_n. \tag{31}$$

$\mu_{R_n,t}$ and $\boldsymbol{\sigma}_{R_n,t}$ denote the expected return and the vector of volatility terms (loadings on the aggregate shocks) of asset n's return, respectively. Notice that the volatilities of return are driven by an exogenous component $\boldsymbol{\sigma}_{n,t}$ and an endogenous component $\boldsymbol{\sigma}_{q_n,t}$, which captures the firm's investment opportunities. Define the vector of expected excess return, $\boldsymbol{\lambda}_{R,t} \equiv \binom{\mu_{R_1,t} - r_t}{\mu_{R_2,t} - r_t}$. The row n of the matrix $\boldsymbol{\Sigma}_{R,t}$ equals $\boldsymbol{\sigma}'_{R_n,t}$. The variance-covariance matrix of the returns is $\boldsymbol{\Omega}_{R,t} = \boldsymbol{\Sigma}_{R,t} \boldsymbol{\Sigma}'_{R,t}$.

4.2.2 Investors' problems

The investor's problem for each type $i \in \{A, B\}$ is to maximize lifetime utility subject to the portfolio risk constraint (13). It can be written recursively as

$$0 = \max_{\substack{c_{i,t} \geq 0 \\ \|\boldsymbol{\sigma}_{i,t}\| \leq \zeta}} \left\{ f_i(c_{i,t} W_{i,t}, V_{i,t}) dt + \mathbb{E}_t[dV_{i,t}] \right\}. \tag{32}$$

With homothetic preferences, the following proposition gives investors' value functions.

Proposition 4 *The value function of investor $i \in \{A, B\}$ has the form*

$$V_{i,t}(W_{i,t}, x_t, w_t) = \frac{W_{i,t}^{1-\gamma_i}}{1-\gamma_i} \xi_{i,t}(x_t, w_t)^{\frac{1-\gamma_i}{1-\psi}}, \tag{33}$$

where $\xi_{i,t}$ captures the type i investor's future investment opportunities. The dynamics of $\xi_{i,t}$ is given by

$$\frac{d\xi_{i,t}}{\xi_{i,t}} = \mu_{\xi_i,t}\, dt + \boldsymbol{\sigma}'_{\xi_i,t}\, d\boldsymbol{B}_t, \tag{34}$$

where $(\mu_{\xi_i,t}, \boldsymbol{\sigma}_{\xi_i,t})$ is determined in equilibrium. The expressions for $\mu_{\xi_i,t}$ and $\boldsymbol{\sigma}_{\xi_i,t}$ are given in the Appendix A.

The HJB equation for the investors' problem is given by the following partial differential equation (PDE):

$$\begin{aligned} 0 = \max_{\substack{c_{i,t}\geq 0 \\ \|\boldsymbol{\sigma}_{i,t}\|\leq \zeta}} \Bigg\{ & \frac{1}{1-1/\psi}\left(c_{i,t}^{1-1/\psi}\xi_{i,t}^{1/\psi} - \rho\right) + \left(\boldsymbol{\sigma}'_{i,t}\boldsymbol{\eta}_t + r_t - c_{i,t}\right) \\ & + \frac{1}{1-\psi}\mu_{\xi_i,t} - \frac{\gamma_i}{2}\boldsymbol{\sigma}'_{i,t}\boldsymbol{\sigma}_{i,t} \\ & + \frac{1}{2}\frac{1}{1-\psi}\left(\frac{1-\gamma_i}{1-\psi} - 1\right)\boldsymbol{\sigma}'_{\xi_i,t}\boldsymbol{\sigma}_{\xi_i,t} \\ & + \frac{1-\gamma_i}{1-\psi}\boldsymbol{\sigma}'_{i,t}\boldsymbol{\sigma}_{\xi_i,t} \Bigg\}. \end{aligned} \tag{35}$$

Proof. See Appendix A. ■

Taking prices and continuation-value dynamics as given, the first-order conditions yield optimal consumption and desired risk exposure. The latter is the exposure the investor would choose without their own portfolio constraint, holding all equilibrium objects fixed in the constrained economy:

$$c_{i,t} = \xi_{i,t}, \tag{36}$$

$$\boldsymbol{\sigma}^*_{i,t} = \frac{1}{\gamma_i}\boldsymbol{\eta}_t + \frac{1}{\gamma_i}\frac{1-\gamma_i}{1-\psi}\boldsymbol{\sigma}_{\xi_i,t}. \tag{37}$$

In equilibrium, the consumption-wealth ratio, which equals $\xi_{i,t}$, responds to changes in investment opportunities. The optimal unconstrained risk exposure has two components, as in the intertemporal capital asset pricing model (ICAPM) result of Merton (1973). The first term in (37) is the myopic demand of a one-period mean-variance investor, and is driven by the market price of risk. The intermediaries' lower risk aversion implies a stronger myopic re-

sponse to changes in the market price of risk. The second term is the hedging demand that captures variation in the investor's investment opportunity set.

Imposing the constraint (13) on the desired exposure (37) yields

$$\boldsymbol{\sigma}_{i,t} = \begin{cases} \boldsymbol{\sigma}^*_{i,t}, & \|\boldsymbol{\sigma}^*_{i,t}\| \le \zeta, \\ \zeta \dfrac{\boldsymbol{\sigma}^*_{i,t}}{\|\boldsymbol{\sigma}^*_{i,t}\|}, & \|\boldsymbol{\sigma}^*_{i,t}\| > \zeta. \end{cases} \tag{38}$$

The portfolio objective in (35) is quadratic in risk exposure with curvature $\gamma_i > 0$, so the constrained optimum is the Euclidean projection of desired exposure onto the ball of radius ζ. When the constraint binds, all components of desired exposure are scaled by a common factor. Prices and continuation values in $\boldsymbol{\sigma}^*_{i,t}$ are those of the equilibrium under consideration.

The following proposition links the constraint on total portfolio risk to the marginal cost of holding each sector's equity.

Proposition 5 *For investor $i \in \{A, B\}$, let $\nu_{i,t} \ge 0$ denote the multiplier on $\zeta - \|\boldsymbol{\sigma}_{i,t}\|$ in the HJB (35). Then*

$$\nu_{i,t} = \gamma_i \max\left\{\|\boldsymbol{\sigma}^*_{i,t}\| - \zeta, 0\right\}, \tag{39}$$

with $\nu_{i,t}(\zeta - \|\boldsymbol{\sigma}_{i,t}\|) = 0$.

For $\|\boldsymbol{\sigma}_{i,t}\| > 0$ and $n \in \{1, 2\}$, consider increasing the weight on sector n's equity, holding wealth and the other risky-asset weight fixed and reducing the risk-free bond weight by the same amount. The marginal contribution to investor i's portfolio risk is

$$m_{in,t} \equiv \frac{\partial \|\boldsymbol{\Sigma}'_{R,t}\boldsymbol{\omega}_{i,t}\|}{\partial \omega_{in,t}} = \frac{\boldsymbol{\sigma}'_{R_n,t}\boldsymbol{\sigma}_{i,t}}{\|\boldsymbol{\sigma}_{i,t}\|}. \tag{40}$$

The investor's asset-level first-order condition is

$$\mu_{R_n,t} - r_t = \gamma_i \boldsymbol{\sigma}'_{R_n,t}\boldsymbol{\sigma}_{i,t} - \frac{1-\gamma_i}{1-\psi}\boldsymbol{\sigma}'_{R_n,t}\boldsymbol{\sigma}_{\xi_i,t} + \chi_{in,t}, \tag{41}$$

where $\chi_{in,t} \equiv \nu_{i,t} m_{in,t}$ is the asset-specific constraint term. Consequently,

$$\chi_{i1,t} - \chi_{i2,t} = \nu_{i,t}(m_{i1,t} - m_{i2,t}). \tag{42}$$

At $\boldsymbol{\sigma}_{i,t} = \mathbf{0}$, the constraint is slack, $\nu_{i,t} = 0$, and $\chi_{in,t} = 0$.

Proof. See Appendix A. ■

The expected excess return (41) is written as the sum of three components: compensation for portfolio risk, an adjustment for hedging value, and the asset-specific cost of the portfolio

constraint. The multiplier measures the marginal value of relaxing the investor's risk limit. Its effect on a particular asset depends on that asset's covariance with the investor's portfolio, as captured by $m_{in,t}$. An asset that reduces portfolio risk has a negative marginal contribution and hence a negative constraint term when $\nu_{i,t} > 0$. A positive multiplier creates different direct constraint costs across sectors precisely when their marginal contributions differ. The portfolio constraint affects expected returns through its effects on equilibrium risk prices and return exposures. The terms $\chi_{in,t}$ identify the investor-specific shadow costs underlying these equilibrium outcomes.

4.3 Markov equilibrium

I define a stationary recursive Markov equilibrium with state variables x_t and w_t, in which all equilibrium objects are expressed as functions of (x, w).

Definition 2 *A Markov equilibrium in state variables (x, w) is a set of functions: equilibrium prices $(r, \boldsymbol{\eta})$, firms' valuation ratio q_n and policy function g_n for $n \in \{1, 2\}$, investors' marginal value of wealth ξ_i and policy functions c_i and $\boldsymbol{\sigma}_i$ for $i \in \{A, B\}$ such that*

i. valuation ratio q_n solves the HJB equation for the firm in sector n, and g_n is the corresponding policy function, taking $(r, \boldsymbol{\eta})$ and laws of motion for x and w as given;

ii. marginal value of wealth ξ_i solves the HJB equation for investor of type i, and c_i and $\boldsymbol{\sigma}_i$ are the corresponding policy functions, taking $(r, \boldsymbol{\eta})$, (q_1, q_2), and laws of motion for x and w as given;

iii. markets clear:

$$w c_A + (1-w) c_B = z \frac{A_1 - \iota_1}{q_1} + (1-z) \frac{A_2 - \iota_2}{q_2} \quad \textit{(goods market)},$$

$$\begin{pmatrix} \boldsymbol{\sigma}_A & \boldsymbol{\sigma}_B \end{pmatrix} \begin{pmatrix} w \\ 1-w \end{pmatrix} = \begin{pmatrix} \boldsymbol{\sigma}_{R_1} & \boldsymbol{\sigma}_{R_2} \end{pmatrix} \begin{pmatrix} z \\ 1-z \end{pmatrix} \quad \textit{(risk exposures market)};$$

iv. the laws of motion for x and w satisfy (18) and (17), respectively.

4.4 Implications from the unconstrained economy

While the equilibrium does not permit closed-form solutions even in the absence of financial constraints, this section presents propositions that characterize risk-taking decisions and asset prices in the unconstrained economy.

4.4.1 Investors' risk-taking decisions

The following proposition characterizes the intermediaries' demand for risk.

Proposition 6 *In the absence of financial constraints, intermediaries' demand for risk from sector 1, i.e., the first element of their risk exposure $\boldsymbol{\sigma}_A^*$, is given by*

$$\sigma_A^{(1),*} = \frac{zR_w + R_w Q_x x(1-x) - R_x(Q_w + \frac{1}{w})x(1-x) - z\frac{\gamma_B}{w(1-w)} - \frac{\gamma_B}{w(1-w)}x(1-x)Q_x}{(\gamma_B - \gamma_A)Q_w - \frac{\gamma_B}{1-w} - \frac{\gamma_A}{w} + R_w}\sigma_1 \tag{43}$$

where

$$\begin{aligned} R_x &= \frac{1-\gamma_A}{1-\psi}\frac{\xi_{A,x}}{\xi_A} - \frac{1-\gamma_B}{1-\psi}\frac{\xi_{B,x}}{\xi_B}; \quad & R_w &= \frac{1-\gamma_A}{1-\psi}\frac{\xi_{A,w}}{\xi_A} - \frac{1-\gamma_B}{1-\psi}\frac{\xi_{B,w}}{\xi_B}; \\ Q_x &= z\frac{q_{1,x}}{q_1} + (1-z)\frac{q_{2,x}}{q_2}; & Q_w &= z\frac{q_{1,w}}{q_1} + (1-z)\frac{q_{2,w}}{q_2}. \end{aligned} \tag{44}$$

The intermediaries' demand for risk from sector 2, i.e., the second element of their risk exposure $\boldsymbol{\sigma}_A^*$ can be found in the Appendix A.

Proof. See Appendix A. ■

Proposition 6 connects investors' risk-sharing incentives to the valuation of productive capital. First, the terms R_x and R_w capture differences in how investors' marginal values of wealth respond to changes in capital composition and wealth distribution, respectively. These differences shape investors' hedging demands and thereby influence how aggregate risks are shared. From (33), the marginal value of wealth is $V_{i,W} \equiv \partial V_i / \partial W_i = W_i^{-\gamma_i}\xi_i^{(1-\gamma_i)/(1-\psi)} > 0$. Holding each investor's own wealth fixed, the risk-sharing terms can therefore be written as

$$R_x = \frac{\partial \log V_{A,W}}{\partial x} - \frac{\partial \log V_{B,W}}{\partial x}, \quad R_w = \frac{\partial \log V_{A,W}}{\partial w} - \frac{\partial \log V_{B,W}}{\partial w}. \tag{45}$$

A positive R_x means that $\log V_{A,W}$ responds more positively to an increase in x than $\log V_{B,W}$; R_w has the analogous interpretation for w. Investors value payoffs especially in states where their marginal value of wealth is high, so differences in these responses create different incentives to hedge changes in investment opportunities. Changing productive composition thus introduces a source of hedging demand through R_x, alongside the wealth distribution channel captured by R_w.

Second, the portfolio revaluation terms Q_x and Q_w capture the value-weighted sensitivities of sectoral valuations to capital composition and wealth distribution, respectively. Under $S_n = K_n q_n$, the price q_n also equals the firm's marginal value of capital. These responses guide firms' investment choices and, together with the diffusion of the state variables, determine the valuation component of aggregate equity risk. Holding each firm's capital stock fixed, the

Q terms can therefore be written as

$$Q_x = z\frac{\partial \log S_1}{\partial x} + (1-z)\frac{\partial \log S_2}{\partial x}, \quad Q_w = z\frac{\partial \log S_1}{\partial w} + (1-z)\frac{\partial \log S_2}{\partial w}. \tag{46}$$

The weights z and $1-z$ are the sectors' shares of total equity value. Thus, $Q_w > 0$ means that an increase in intermediary wealth share raises valuations on a value-weighted basis, even if the two sectors respond in opposite directions. The term Q_x captures the corresponding valuation response to capital composition, holding firm capital stocks fixed. Through (31), these valuation responses change the risks carried by the aggregate equity portfolio. Their interaction with investors' hedging incentives links equilibrium risk sharing to the valuation of productive capital, which in turn guides sectoral investment through (25).

It is worth noting that Proposition 6 nests the case of a single productive sector studied in the literature. When there is only one sector (i.e., one Brownian shock), z equals unity, and Q_x and R_x disappear. In this case, there is no capital reallocation, as investors do not have an option to shift risk exposure across sectors, and the risk sharing is driven solely by the wealth distribution channel.[14]

Corollary 1 *Risk is concentrated in intermediaries (i.e., $\omega_A^{(n),*} > \omega_B^{(n),*}$ for $n \in \{1,2\}$), if and only if $\gamma_A < \gamma_B$.*

A corollary of Proposition 6 is that intermediaries invest more of their wealth in both risky assets, and therefore bear more risks than households. Effectively, intermediaries act not only as risk 'bearers', but also as risk 'allocators' in the economy. As will be shown below, the risk pricing in the financial markets determines the capital allocation across sectors. In the absence of financial frictions, i.e., intermediaries are freely sharing risks with households, efficiency in financial markets and capital allocation is achieved simultaneously.

4.4.2 Asset pricing implications

The following propositions present the expressions for the market price of risk $\boldsymbol{\eta}_t$ and the interest rate r_t, which highlight the key implications of asset pricing.

Proposition 7 *In the absence of financial constraints, the market price of risk $\boldsymbol{\eta}_t$ can be ex-*

[14] The risk exposure of intermediaries reduces to $\sigma_A^* = \frac{R_w - \frac{\gamma_B}{w(1-w)}}{(\gamma_B-\gamma_A)\frac{q_w}{q} - \frac{\gamma_B}{1-w} - \frac{\gamma_A}{w} + R_w}\sigma$.

pressed as following

$$\begin{aligned}\boldsymbol{\eta}_t = \gamma_t \left[z_t \boldsymbol{\sigma}_{R_1,t} + (1-z_t)\boldsymbol{\sigma}_{R_2,t} \right] \\ - \gamma_t \left[w_t \left(\frac{1}{\gamma_A}\right)\left(\frac{1-\gamma_A}{1-\psi}\right)\boldsymbol{\sigma}_{\xi_A,t} + (1-w_t)\left(\frac{1}{\gamma_B}\right)\left(\frac{1-\gamma_B}{1-\psi}\right)\boldsymbol{\sigma}_{\xi_B,t} \right],\end{aligned} \tag{47}$$

where γ_t denotes aggregate risk aversion, and is given by

$$\gamma_t = \left(\frac{w_t}{\gamma_A} + \frac{1-w_t}{\gamma_B} \right)^{-1}. \tag{48}$$

Proof. See Appendix A. ■

The market price of risk (47) is the product of aggregate risk aversion and the difference between the supply of risk and the average hedging demand.[15] Aggregate risk aversion (48) is the reciprocal of the wealth-weighted average of the risk tolerance of the two types of investors. Hence, periods during which intermediaries are relatively less capitalized, so aggregate risk aversion is high, tend to experience a high market price of risk. This is consistent with the heterogeneous agents asset pricing literature (Panageas, 2020). The second term, wealth-weighted average of marginal value of wealth, with the weight also accounting for differences in preferences, captures the idea that if the average hedging demand is high, then a lower market price of risk is required to induce investors to bear the risk.

The vector of risk premia of the two assets is given by

$$\boldsymbol{\lambda}_{R,t} = \boldsymbol{\Sigma}_{R,t}\boldsymbol{\eta}_t. \tag{49}$$

Following the expression for market price of risk, the risk premia consist of two terms as well. The first term is driven by the covariance between each risky asset and total wealth. Intuitively, an asset that comoves a lot with total wealth provides little diversification benefit, is therefore risky, and commands a high risk premium. The second term captures the fact that an asset whose payoffs are high when the average hedging demand is high provides a good hedge to investors and therefore requires a lower risk premium.

Proposition 8 *In the absence of financial constraints, the interest rate r_t can be expressed as following*

$$\psi\rho + \mu_t + (1-\psi)(r_t + \Phi_t) = z_t \frac{A_1 - \iota_1(g_{1,t})}{q_{1,t}} + (1-z_t)\frac{A_2 - \iota_2(g_{2,t})}{q_{2,t}}, \tag{50}$$

[15] The term $z_t\boldsymbol{\sigma}_{R_1,t} + (1-z_t)\boldsymbol{\sigma}_{R_2,t}$ is the value-weighted average of the diffusion of the two risky assets, which is also the diffusion of total wealth.

where $\mu_t = w_t \mu_{\xi_A,t} + (1-w_t)\mu_{\xi_B,t}$, *and* $\Phi_t = w_t \Phi_{A,t} + (1-w_t)\Phi_{B,t}$. *The expressions for* $\mu_{\xi_i,t}$ *and* $\Phi_{i,t}$ *are given in the Appendix A.*

Proof. See Appendix A. ■

The left-hand side of (50) represents the (normalized) aggregate demand for goods, while the right-hand side represents the (normalized) aggregate supply of goods. The effects of the interest rate on aggregate demand depends crucially on the EIS: if $\psi > 1$, then an increase in the interest rate decreases aggregate demand (the opposite is true when $\psi < 1$).

4.5 Risk pricing and capital allocation

The following corollary collects the pricing and investment conditions that link financial markets to sectoral capital allocation in both constrained and unconstrained equilibria.

Corollary 2 *In equilibrium, the expected return on each sector's equity, for* $n \in \{1,2\}$, *satisfies*

$$\frac{A_n - \iota_{n,t}}{q_{n,t}} + g_{n,t} + \mu_{q_n,t} + \boldsymbol{\sigma}'_{q_n,t}\boldsymbol{\sigma}_n = r_t + \boldsymbol{\eta}'_t(\boldsymbol{\sigma}_{q_n,t} + \boldsymbol{\sigma}_n), \tag{51}$$

which can also be written as

$$\mu_{R_n,t} = r_t + \boldsymbol{\eta}'_t \boldsymbol{\sigma}_{R_n,t}. \tag{52}$$

In both constrained and unconstrained equilibria, (51) equates the expected return on sectoral equity to the risk-free rate plus compensation for its exposure to aggregate shocks. The investment condition (25) connects equilibrium valuations to capital accumulation. Under the AK technology, sectoral marginal products of capital are fixed at A_n, so their dispersion alone does not identify capital misallocation, consistent with the risk-adjusted perspective of David et al. (2022). Endogenizing the SDF allows capital allocation to influence the pricing of risk. In the unconstrained benchmark, Proposition 7 shows how wealth-weighted risk tolerance, the composition of aggregate wealth, and investors' hedging demand jointly determine risk prices.

Financial constraints introduce an additional channel through the scarcity and allocation of risk-bearing capacity. Proposition 5 identifies the marginal constraint cost of shifting portfolio weight from sector 2 to sector 1 as $\nu_{i,t}(m_{i1,t} - m_{i2,t})$: the multiplier measures the investor's private shadow cost of risk-bearing capacity, while marginal risk contributions determine its asset-specific incidence. These constraint costs affect sectoral investment through equilibrium risk prices and valuations. Investment changes capital composition x_t, while portfolio returns and consumption choices change intermediary wealth share w_t, feeding back into the risks investors must absorb and their capacity to bear them. Section 5 evaluates the resulting

investment distortions relative to the unconstrained equilibrium at the same state (x, w) and their implications for capital-composition dynamics.

4.6 Alternative economies

Although the model is analytically tractable, one may wonder how each segment of the model contributes to the equilibrium and if the full setup is necessary to capture the key mechanisms. In this section, I consider alternative economies that can provide intuition in this regard. I rely on the method of asymptotic expansion, which yields semi-closed form solutions for the equilibrium objects of these economies. This allows to shed light on the roles of each segment of the model before proceeding to numerical solutions.

The first (frictionless) economy features one representative sector, and no heterogeneity in risk aversion among investors. The following lemma shows that up to scale, there are no fluctuations in the equilibrium objects, such as investment, consumption, and asset prices.

Lemma 1 *Consider an economy where $\gamma_A = \gamma_B = \gamma$ and two sectors are identical. Under these assumptions, the structure of aggregate shocks simplifies to only one Brownian shock, with volatility σ, i.e.,*

$$\frac{dK_t}{K_t} = g_t\,dt + \sigma\,dB_t. \tag{53}$$

The following equilibrium objects can be derived.

i. The consumption-wealth ratio and risk exposure are given by

$$c_A^* = c_B^* = \rho - (1-\psi^{-1})\left(g^* - \frac{\gamma\sigma^2}{2}\right), \qquad \sigma_A^* = \sigma_B^* = \sigma. \tag{54}$$

ii. The investment rate and price of capital satisfy

$$\frac{A-\iota^*}{q^*} = \rho - \left(1-\psi^{-1}\right)\left(g^* - \frac{\gamma\sigma^2}{2}\right), \qquad g'(\iota^*) = \frac{1}{q^*}. \tag{55}$$

iii. The market price of risk and interest rate are given by

$$\eta^* = \gamma\sigma, \qquad r^* = \rho + \psi^{-1}g^* - (1+\psi^{-1})\frac{\gamma\sigma^2}{2}. \tag{56}$$

Proof. See Appendix A. ■

The above lemma shows that in the absence of heterogeneity in risk aversion, the one-sector economy is essentially deterministic. Output, consumption, and investment all move

in proportion to the capital stock, which is subject to aggregate shocks. Since investment opportunities are constant in this case, investors maintain a constant portfolio weight of one over time. The market price of risk and the interest rate are both constant as well.

Consider next the second (frictionless) economy featuring two productive sectors and no heterogeneity in risk aversion among investors, à la Eberly and Wang (2025). Even though a closed-form solution is not available in this case, following Silva (2020), the behavior of the economy can be characterized by considering an asymptotic expansion of the equilibrium around the economy in Lemma 1.[16]

Lemma 2 *Consider a parametric family of economies indexed by ε, where two sectors are not identical. Assume $\gamma_A = \gamma_B = \gamma$. In addition, assume*

$$\frac{dK_{1,t}}{K_{1,t}} = g_{1,t}dt + \sigma\varepsilon dB_{1,t} + \sigma(1-\varepsilon)dB_{2,t}, \tag{57}$$

$$\frac{dK_{2,t}}{K_{2,t}} = g_{2,t}dt + \sigma dB_{2,t}, \tag{58}$$

and productivity A, depreciation rate δ, and adjustment cost (ϕ,θ) are assumed to be the same across two sectors for expositional simplicity.

The equilibrium objects, as functions of x, can be solved up to the first order in ε.

i. The prices of capital in sector 1 is given by

$$q_1(x;\varepsilon) = q^* + \frac{q^*\gamma\sigma^2(-c^*x\theta + c^*\theta\psi - q^*x + q^*)}{c^*(c^*\theta\psi + q^*)}\varepsilon + O(\varepsilon^2). \tag{59}$$

ii. The market prices of risks are given by

$$\eta_1(x;\varepsilon) = \gamma\sigma z\varepsilon + O(\varepsilon^2), \qquad \eta_2(x;\varepsilon) = \gamma(\sigma - \sigma z\varepsilon) + O(\varepsilon^2). \tag{60}$$

where z is the weight of sector 1 in the market portfolio, i.e., $z = \left(1 + \frac{q_2}{q_1}\frac{1-x}{x}\right)^{-1}$.

All other equilibrium objects can be found in Appendix A.

Proof. See Appendix A. ■

The above lemma highlights that the introduction of a second sector leads to the equilibrium objects becoming functions of the sectoral composition of the economy, x. The sectoral allocation is always efficient in risk-adjusted terms, as the representative investor frictionlessly

[16] Note that sectoral investment rate is monotonically increasing in its sectoral Tobin's Q, hence they convey essentially the same information for the given adjustment cost function.

reallocates capital in response to shocks.[17] There is, however, no role of frictions in the financial markets, such as the one considered in the full model (i.e., margin constraints), in generating capital misallocation, because the representative investor absorbs all the risks and does not have the capacity to share risks.

Consider next the third (frictionless) economy featuring one sector and heterogeneous investors. The behavior of the economy can again be characterized by considering an asymptotic expansion of the equilibrium around the economy in Lemma 1, as no closed-form solution is available in this case either.

Lemma 3 *Consider a parametric family of economies indexed by* ϵ*, where two sectors are identical. The structure of aggregate shocks boils down to only one Brownian shock, with volatility* σ*, as in Lemma 1. Let* $\gamma_A = \gamma(1-\Delta\gamma)$*,* $\gamma_B = \gamma(1+\Delta\gamma)$*,* $\Delta\gamma \equiv \delta_\gamma \epsilon$*, and* $\delta_\gamma > 0$*.*

The equilibrium objects, as functions of w*, can be solved up to the first order in* ϵ*.*

i. Risk exposures

$$\sigma_A(w;\epsilon) = \sigma + 2(1-w)\sigma\Delta\gamma + O(\epsilon^2), \tag{61}$$

$$\sigma_B(w;\epsilon) = \sigma - 2w\sigma\Delta\gamma + O(\epsilon^2). \tag{62}$$

ii. Market price of risk

$$\eta(w;\epsilon) = \gamma\sigma - \gamma\sigma(2w-1)\Delta\gamma + O(\epsilon^2). \tag{63}$$

iii. The growth rate of capital

$$g(w;\epsilon) = g^* + \frac{(\psi-1)q^*}{q^* + \phi_1\psi c^*}\frac{\gamma\sigma^2}{2}(2w-1)\Delta\gamma + O(\epsilon^2). \tag{64}$$

All other equilibrium objects can be found in Appendix A.

Proof. See Appendix A. ■

The above lemma, in comparison to Lemma 1, showcases the role of heterogeneity in risk aversion in driving fluctuations in the equilibrium. As in Proposition 6, intermediaries (type A) are more exposed to risk than households (type B), i.e., $\sigma_A(w) > \sigma_B(w)$ for $w \in (0,1)$. Financial intermediaries operate leveraged, and after a negative shock, their wealth share declines, leading to an increase in the market price of risk and, provided $\psi > 1$, a fall in investment and growth.[18] This mechanism, often referred to as the *balance-sheet channel* in the macro-finance literature (Di Tella, 2017), is a result of risk sharing among investors, which makes the

[17] Appendix C presents the global solution of this economy.

[18] The condition $\psi > 1$ guarantees that the price of the risky asset falls after a negative shock, as the increase in the risk premium dominates the decline in interest rates.

intermediaries' wealth share, w_t, a state variable that drives time-variation in risk premia.[19] There is no capital reallocation or misallocation in this economy, as the investors can only trade one risky asset, and the sectoral composition of the economy is constant.

The fourth economy introduces the financial constraints in the form of (13) (applied to one risky asset) on top of the setup in Lemma 3.

Lemma 4 *Consider a parametric family of economies indexed by ϵ, where two sectors are identical. The structure of aggregate shocks boils down to only one Brownian shock, with volatility σ, as in Lemma 1 and 3. Let $\gamma_A = \gamma(1-\Delta\gamma)$, $\gamma_B = \gamma(1+\Delta\gamma)$, $\Delta\gamma \equiv \delta_\gamma \epsilon$, and $\delta_\gamma > 0$. Moreover, investors are subject to the financial constraints. Specifically, $\sigma_i \leq \bar{\zeta}$.*

In equilibrium, when the constraint does not bind, σ_A can be expressed as (61) from Lemma 3. Other equilibrium objects behave like the ones in Lemma 3.

When the constraint binds, σ_A equals to $\bar{\zeta}$, and the equilibrium objects can be solved up to the first order in ϵ as follows.

i. Risk exposure for investors of type B

$$\sigma_B(w;\epsilon) = \frac{\sigma - w\bar{\zeta}}{1-w} + O(\epsilon^2). \tag{65}$$

ii. Market price of risk

$$\eta(w;\epsilon) = \gamma(1+\Delta\gamma)\frac{\sigma - w\bar{\zeta}}{1-w} + O(\epsilon^2). \tag{66}$$

iii. The growth rate of capital

$$g(w;\epsilon) = g^* + q^*(\psi-1)\frac{2w^2\bar{\zeta}^2 - w\bar{\zeta}^2 - 2w\bar{\zeta}\sigma + \sigma^2}{(g^*\theta\psi - q^* - r^*\theta\psi)(1-w) + w\gamma\bar{\zeta}\theta\sigma\psi - \gamma\theta\psi\sigma^2}\frac{\gamma}{2}\Delta\gamma + O(\epsilon^2). \tag{67}$$

All other equilibrium objects can be found in Appendix A.

Proof. See Appendix A. ■

The first observation is that in equilibrium, financial constraints will bind for intermediaries, when their wealth share w is sufficiently low. This is because the unconstrained risk exposure of intermediaries (61) is higher than that of households, and it increases as w decreases. Second, when the constraint binds, more risk-averse households become marginal investors. The risk exposure of households (65) increases relative to the unconstrained economy, as intermediaries are forced to sell assets. Importantly, the market price of risk increases

[19] Appendix C presents the global solution of this economy.

relative to the unconstrained economy from Lemma 3, due to limited risk sharing among investors. The inefficient financial market in this economy leads to intertemporally distorted investment, and no capital misallocation arises in this economy.

These simplified economies distinguish changes in aggregate investment from changes in sectoral allocation. Lemma 2 isolates the role of capital composition in a representative-investor economy. Lemmas 3 and 4 show how intermediary wealth and constrained risk sharing affect investment when sectoral composition is fixed. In the full model, intermediary wealth and capital composition jointly determine portfolio choices, risk prices, and whether the risk constraint binds. Its incidence across sectoral claims depends on their marginal contributions to portfolio risk. The resulting investment responses change capital composition and hence the risks investors must share in the future.

5 Quantitative analysis

This section examines how intermediary wealth and capital composition jointly shape sectoral investment. I solve the model's system of PDEs numerically using a global solution method based on neural networks. I first describe the numerical method and calibration. I then discuss the unconstrained benchmark. Finally, I compare constrained and unconstrained equilibria to study investment distortions and their implications.

5.1 Numerical solution

The model is analytically tractable: the equilibrium dynamics can be fully characterized by a system of PDEs. The computation of equilibrium requires solving the HJB equations of firms and investors simultaneously. The value functions $q_n(x,w)$ for $n \in \{1,2\}$ and $\xi_i(x,w)$ for $i \in \{A,B\}$ can be found by solving a system of second-order PDEs in (x,w). To do so, all equilibrium objects need to be expressed in terms of these value functions and their derivatives. Unfortunately, the system of nonlinear differential equations does not admit a closed-form solution, so I rely on numerical techniques. This is challenging in the presence of the model's two endogenous state variables and financial frictions that occasionally bind.

I extend the global projection methods (Judd, 1992, 2023) using neural networks to approximate the value functions. Unlike a log-linearized representation around the steady state, this global method provides a full characterization of the whole dynamic system. This is particularly important as the model features regions in which financial constraints occasionally bind and the dynamics are highly non-linear. This is crucial for understanding the joint dynamics of (imperfect) risk sharing and capital misallocation. In practice, I use feedforward neural net-

works and experiment with various architectures, such as the number of hidden layers and the choice of activation functions. For the purposes of this paper, the neural network can be seen as a function approximator, like Chebyshev polynomials. In Appendix B, I discuss the algorithm used to solve the system of PDEs.

5.2 Calibration

Table 1 reports the calibration of the model. Note that the values correspond to annual values as the model is cast in continuous time.

Table 1: Calibration

Parameters	Symbol	Values
Rate of time preference	$\hat{\rho}$	0.001
Mortality rate	κ	0.02
Elasticity of intertemporal substitution	ψ	1.5
RRA of type A investors	γ_A	3
RRA of type B investors	γ_B	18
Population share of type A investors	$\bar{u}$	0.1
Productivity of capital in sector 1	A_1	0.30
Productivity of capital in sector 2	A_2	0.25
Volatility of capital in sector 1	σ_1	0.036
Volatility of capital in sector 2	σ_2	0.03
Depreciation rate of capital in sector 1	δ_1	0.016
Depreciation rate of capital in sector 2	δ_2	0.014
Adjustment cost of capital in sector 1	θ_1	14
Adjustment cost of capital in sector 2	θ_2	15
Tightness of the margin constraint	ζ	0.12

This table reports the parameter values used in calibrating the model. The values correspond to annual values as the model is cast in continuous time.

The rate of time preference $\hat{\rho}$ and the mortality rate κ are set to 0.001 and 0.02, respectively, implying an effective discount rate of $\hat{\rho} + \kappa = 0.021$, and a real interest rate around 2% in annual terms.[20] I set EIS to $\psi = 1.5$ for all investors, a common value found in the asset pricing and macro-finance literature, to focus on the heterogeneity in risk bearing capacity. The RRA of intermediaries (type A) is $\gamma_A = 3$, while households (type B) have $\gamma_B = 18$, preserving higher risk tolerance among intermediaries and investors' preferences for early resolution of uncertainty ($\gamma_i > 1/psi$). Other values within this range generate similar results. To stabilize

[20]For instance, the effective discount rate is close to that in Gârleanu and Panageas (2015).

intermediaries' share of wealth at moderate levels, the population share of type A investors is $\bar{u} = 0.1$. The tightness of the margin constraint is set as $\zeta = 0.12$, which indicate that the bound on the total risk exposure is 0.12, consistent with Kargar (2021).

The model distinguishes the two sectors along two dimensions: productivity and fundamental risk exposure. The sectoral productivities are set as $A_1 = 0.3$ and $A_2 = 0.25$ to match a capital-output ratio of around 4 to 5. The sectoral volatilities are set as $\sigma_1 = 0.036$ and $\sigma_2 = 0.03$. These values are consistent with Assumption 1 that sector 1 is riskier and more productive, representing the growth sector. The values of the volatilities are broadly consistent with the volatility of output growth. The depreciation rates δ_1 and δ_2 are set to 0.016 and 0.014, and the adjustment cost parameters θ_1 and θ_2 are set to 14 and 15, which indicate the relative higher depreciation and lower adjustment cost in the growth sector. These parameter choices are broadly consistent with the macro-finance literature (Brunnermeier and Sannikov, 2014; Silva, 2020; Hansen et al., 2024).

Table 2 reports empirical benchmarks for comparison with simulated model moments. Note that the risk premium is on the value-weighted portfolio of the two risky assets, which has relatively low volatility (around 3%). By comparison the volatility of equity return is around 16.2% in the data. Therefore, the equity premium implied is about five times larger than that of the equity claims, putting it in the range of standard estimates in the literature. This can be seen from the Sharpe ratio as well, which is around 0.43 in the data and 0.35 in the model.

Table 2: Empirical benchmark moments

Moment	Data	Model
Mean output growth (%)	1.9	2.1
Standard deviation of output growth (%)	2.6	2.8
Mean consumption growth (%)	2.1	2.0
Standard deviation of consumption growth (%)	3.2	3.3
Mean wealth-consumption ratio	87	70
Mean risk premium (%)	6.3	1.7
Standard deviation of market return (%)	16.2	3.5
Mean real risk-free rate (%)	1.8	2.3
Standard deviation of real risk-free rate (%)	0.9	0.8
Equity Sharpe ratio	0.43	0.38
Mean price–dividend ratio	32	26
Standard deviation of log price–dividend ratio	0.43	0.33

Macroeconomic moments are calculated directly from BEA annual data via FRED. Output, and consumption denote real GDP, and total personal consumption expenditures. Financial moments are reported data estimates from the literature.

5.3 Unconstrained benchmark

I now discuss the equilibrium in which portfolio risk constraints are absent, obtained by setting ζ to infinity. This section focuses on risk-adjusted capital allocation in an economy where sectoral productivities differ permanently under the AK technology. Risk-averse investors require compensation for exposure to aggregate shocks, so sectoral valuations and investment reflect both productivity and risk. This unconstrained equilibrium provides the benchmark for evaluating how financial constraints affect capital allocation.

The general-equilibrium results highlight the origins of risk compensation and the interaction between the two state variables, capital composition x and intermediary wealth share w, which shapes sectoral investment, asset prices, and investors' risk-taking. In particular, diversification interacts with the *balance-sheet* channel, so the response of sectoral valuations to intermediary wealth varies with capital composition. For sector 1, this response can change sign when its share of productive capital is sufficiently small. The model also generates an endogenous flight-to-safety mechanism: when intermediary capitalization is weak, the relative valuation and investment of the safer sector can rise.

5.3.1 Sectoral dynamics

Figure 1 shows the sectoral investment rates and prices of capital (Tobin's Q) as functions of the state variables x and w. Sectoral outcomes are nonlinear in both state variables. In particular, the response of sectoral investment to intermediary wealth w depends on capital composition x. Equilibrium required returns enter valuations through (51), and the investment condition (25) implies that investment rates move in the same direction as Tobin's Q.[21] I therefore focus on sectoral valuations in the discussion that follows.[22]

Holding intermediary wealth share w fixed, the price of capital in sector 1 tends to be high when its capital share x is low, as shown in the bottom-left panel of Figure 1. Similarly, the price of capital in sector 2 tends to be high when x is high. Claims on the smaller sector provide diversification benefits, supporting a higher valuation. Since sectoral marginal products are fixed at A_n, these valuation patterns reflect the pricing of productive risks and investment opportunities. Such reallocation is key to the survival of both productive sectors, and changes productive composition through capital accumulation.

When either sector dominates productive capital, its valuation tends to increase with intermediary wealth share w, consistent with the *balance-sheet* channel. Lower intermediary

[21] This is consistent with the intuition from David et al. (2022), who derive a risk-adjusted Euler equation in a frictionless neoclassical model. I focus on the GE implications of such risk-adjusted capital allocation.

[22] Sectoral capital growth rates exhibit the same qualitative patterns, since capital growth is increasing in investment on the admissible domain.

capitalization shifts the wealth distribution toward more risk-averse households, raising aggregate risk aversion and lowering the dominant sector's valuation. For some capital compositions, weaker intermediary capitalization lowers both sectoral valuations, with sector 1's valuation declining proportionately more. In other regions, q_1 falls while q_2 rises. Both patterns increase the relative valuation of the safer sector, reflecting an endogenous flight-to-safety mechanism.[23]

When sector 1 is small, diversification can reverse its valuation response to intermediary wealth: Figure 1 contains regions with low x in which q_1 decreases with w. This pattern is consistent with diversification becoming more valuable when wealth is held predominantly by more risk-averse households. Although sector 1 has higher fundamental volatility, its claims provide diversification when it accounts for a small share of productive capital. The sign reversal illustrates the economic role of jointly determining the SDF and capital allocation: the effect of intermediary capitalization on sectoral valuations depends on the composition of the risks being priced.

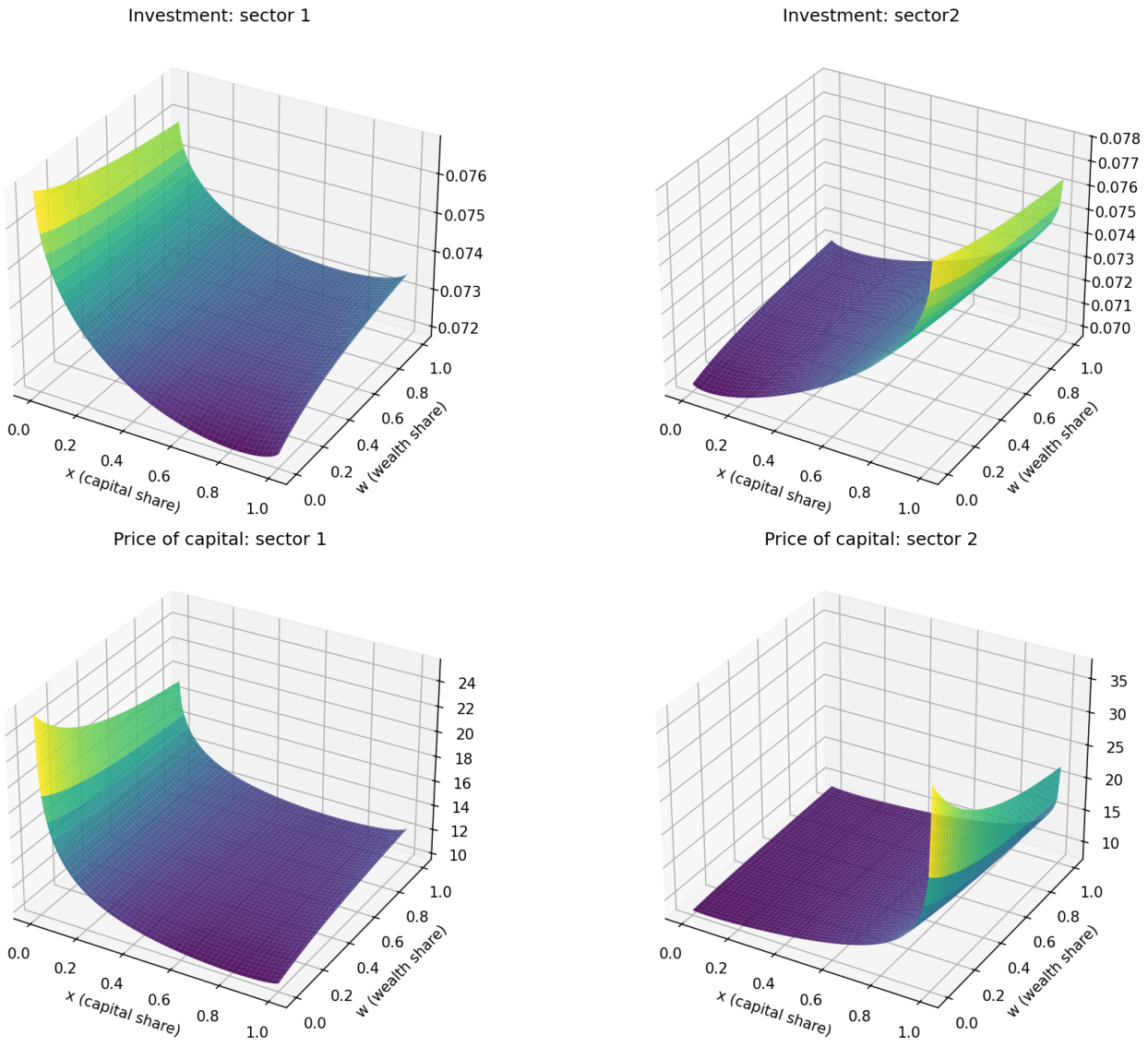
5.3.2 Asset pricing

Figure 2 presents the market prices of risk, η_1 and η_2, as functions of the state variables under the unconstrained calibration. Holding capital composition x fixed, both market prices of risk decrease with intermediary wealth share w. When intermediary capitalization is weak, a larger share of wealth belongs to more risk-averse households, raising aggregate risk aversion (48). Investors require greater compensation per unit of exposure to the fundamental shocks in these states. This dependence of risk compensation on intermediary capitalization reflects the *balance-sheet* channel. The movements in prices also feed back into intermediary wealth: losses on leveraged equity positions can reduce intermediaries' wealth share, further lowering the asset prices. This financial amplification mechanism is familiar from the macro-finance literature (Brunnermeier and Sannikov, 2014).

Although each sector's capital is directly exposed to only one fundamental shock (Assumption 1), equilibrium valuations expose its equity returns to both shocks. The market prices of risk therefore affect each sector through its return exposures (49). The return loadings also vary with the state variables. Consequently, the valuation responses described above depend on how changes in prices of risk interact with each sector's equilibrium exposures, together with changes in the risk-free rate and future investment opportunities. Equilibrium risk pricing shapes capital reallocation through its effects on relative sectoral valuations and investment.

[23] In the model, the flight-to-safety mechanism comes from the endogenous risk-sharing among heterogeneous investors with recursive preferences, instead of exogenously time-varying risk aversion (Kozak, 2022) or habit utility (Duarte, 2018) in a representative investor framework.

Figure 1: Sectoral dynamics

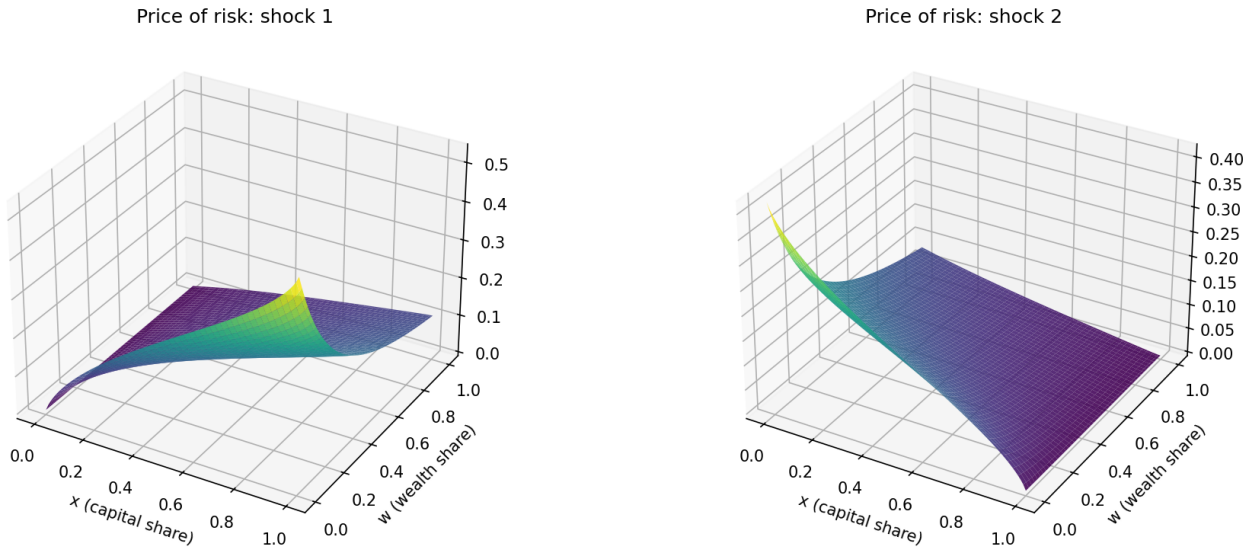


This figure shows the sectoral investment rates and prices of capital (Tobin's Q) as functions of state variables x (capital share of sector 1) and w (wealth share of intermediaries) under the unconstrained calibration. The top panels present the investment rates of the two sectors, and the bottom panels present the prices of capital of the two sectors.

Holding intermediary wealth share w fixed, η_1 increases with capital share x, while η_2 decreases. As sector 1 accounts for more productive capital, its fundamental shock becomes more prevalent in aggregate capital fluctuations. The higher price of the first shock is consistent with investors requiring greater compensation for bearing this risk. At the same time, the second shock becomes less prevalent, and claims on sector 2 offer diversification benefits. The opposite pattern appears as x declines. Thus, productive composition affects the relative prices of the two fundamental risks through the risks investors must absorb and their opportunities to diversify.

The sensitivity of both market prices of risk to capital composition is greater when intermediary wealth share w is low. Changes in the composition of productive risks therefore have larger pricing effects when wealth is held predominantly by more risk-averse households. Proposition 7 connects this interaction to wealth-weighted risk tolerance, aggregate wealth exposures, and investors' hedging demand. These results illustrate the economic role of endogenizing the SDF: capital composition affects the risks being priced, while wealth distribution shapes investors' capacity to bear them. Investment and portfolio returns, in turn, change capital composition and intermediary wealth, feeding back into risk pricing.

Figure 2: Prices of risk

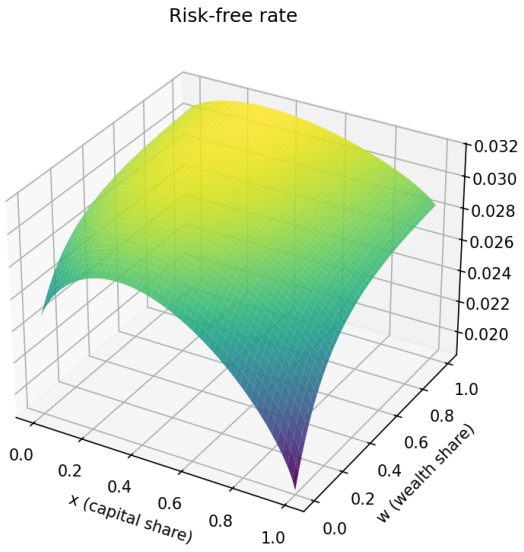


This figure shows the prices of risk as functions of state variables x (capital share of sector 1) and w (wealth share of intermediaries) under the unconstrained calibration. The left panel presents the price of risk for shock B_1, and the right panel presents the price of risk for shock B_2.

Figure 3 presents the risk-free rate, which enters the required return on equity alongside risk compensation in (51). The rate generally increases with intermediary wealth share w.

Lower rates at low w are consistent with stronger precautionary saving when more risk-averse households hold a larger share of wealth. The rate is also hump-shaped in capital composition x, consistent with weaker precautionary saving when the composition of productive capital provides greater diversification benefits. These interest-rate movements accompany the changes in risk prices and jointly affect sectoral valuations.

Figure 3: Risk-free rate



This figure shows the risk-free rate as functions of state variables x (capital share of sector 1) and w (wealth share of intermediaries) under the unconstrained calibration.

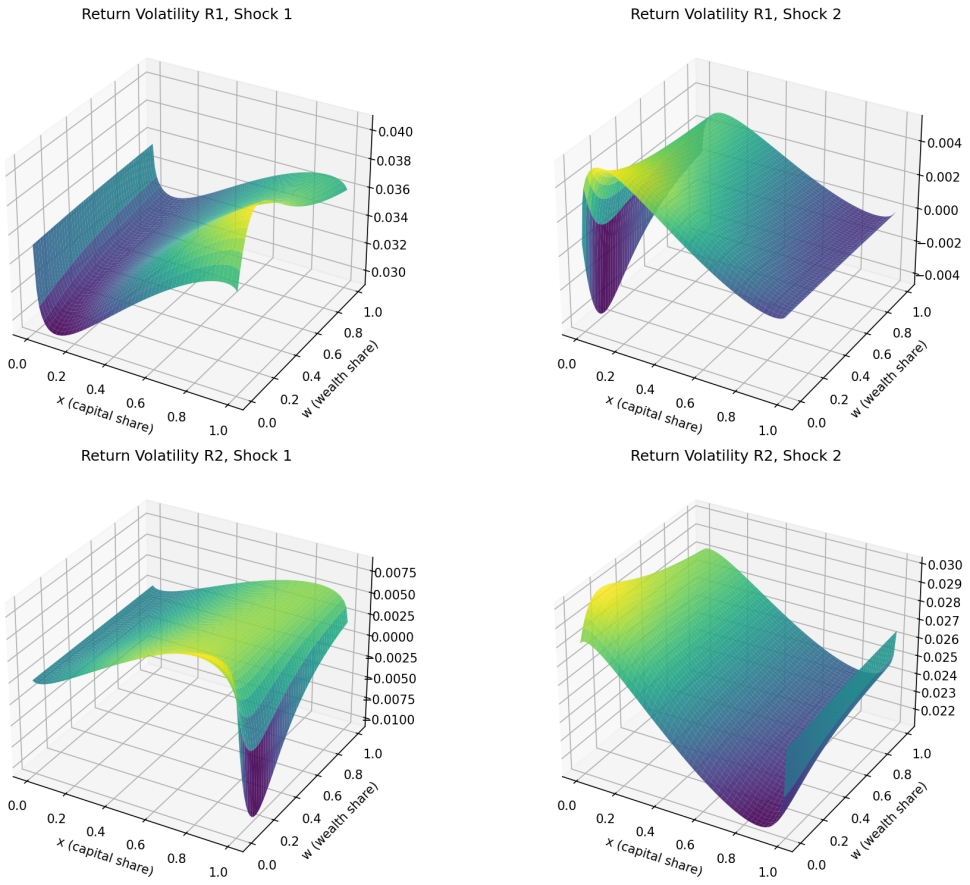
5.3.3 Hedging and financial amplification

With two risky assets, sectoral valuation responses can cushion or reinforce the feedback between asset prices and intermediary wealth. An adverse shock to sector 1 can generate valuation gains on sector 2, partially offsetting intermediaries' losses on their sector 1 holdings. The second asset then provides an endogenous hedge against that shock: its protective return arises through equilibrium valuation adjustments, while it remains exposed to its own fundamental risk. In other states, the same shock lowers both sectoral valuations, transmitting losses across assets through their common dependence on risk sharing.

Figure 4 shows the signed loadings of sectoral returns on the two fundamental shocks. Let $\sigma_{R_n}^{(j)}$ denote sector n's return loading on shock j. The bottom-left panel contains regions with a large sector 1 capital share and low intermediary wealth in which $\sigma_{R_2}^{(1)} < 0$. In these states, an adverse sector 1 shock generates a positive instantaneous return on sector 2. This return response incorporates the shock's effects on both x and w, so it identifies a hedge against the

shock itself. Where $\sigma_{R_2}^{(1)} > 0$, an adverse sector 1 shock instead generates a loss on sector 2, even though its capital has no direct exposure to that shock. The changing sign of this loading shows how equilibrium valuations can generate either hedging or contagion across sectors.

Figure 4: Sectoral return exposures

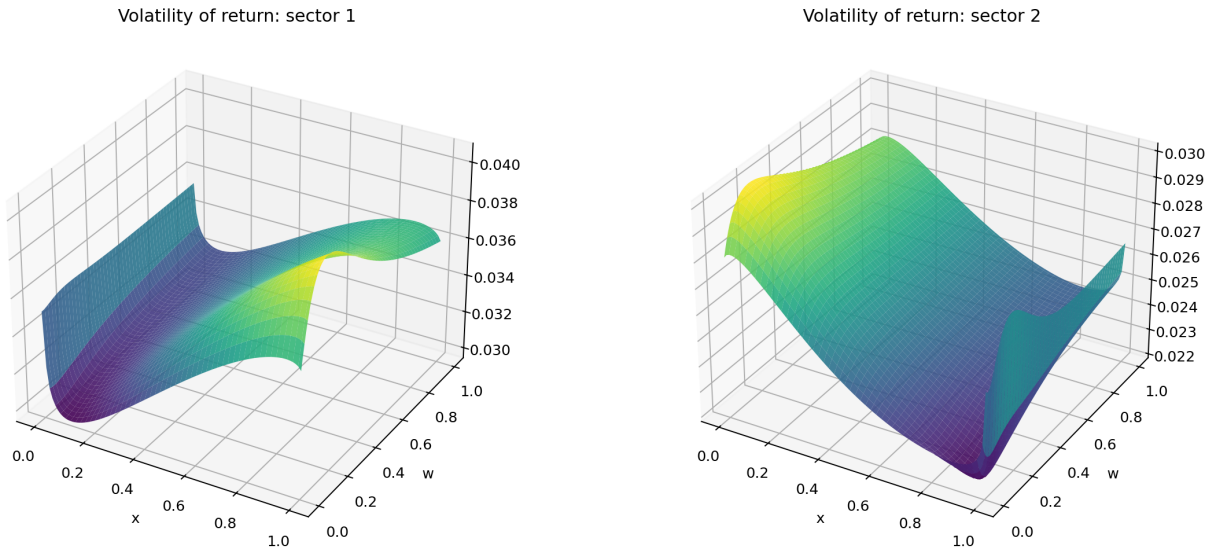


This figure shows the signed return loadings $\sigma_{R_n}^{(j)}$ under the unconstrained calibration as functions of capital composition x and intermediary wealth share w. The top and bottom rows correspond to sectors 1 and 2, respectively; the left and right columns correspond to shocks B_1 and B_2. A negative loading implies a positive instantaneous return response to a negative realization of the corresponding shock.

Figure 5 reports total return volatility, calculated as $\|\boldsymbol{\sigma}_{R_n}\|$, the square root of the sum of squared return loadings. By (31), returns combine the sector's direct fundamental exposure with the response of its valuation to both shocks. Return volatility is non-monotonic in capital composition. For each sector, volatility rises when its capital share becomes very small, where valuations are sensitive to changes in x. At other capital compositions, this valuation response can offset part of the direct fundamental exposure. The resulting volatility patterns reflect the interaction of diversification and the *balance-sheet* channel through equilibrium prices.

To connect these return responses to intermediary capitalization, consider the diffusion of

Figure 5: Volatility of returns



This figure shows the volatility of returns as functions of state variables x (capital share of sector 1) and w (wealth share of intermediaries) under the unconstrained calibration. The left panel presents the volatility of returns for sector 1, and the right panel presents the volatility of returns for sector 2.

wealth share in (17). Denote the coefficient multiplying dB_j in dw by $b_{w,j}$. Using the portfolio representation of investors' risk exposures gives

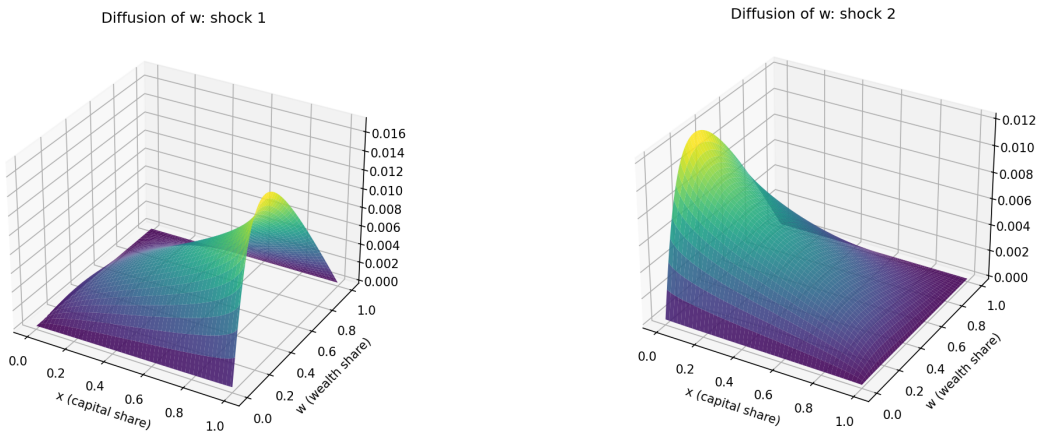
$$b_{w,j} \equiv w(1-w)\sigma_w^{(j)} = w(1-w)\sum_{n=1}^{2}\left(\omega_A^{(n)} - \omega_B^{(n)}\right)\sigma_{R_n}^{(j)}, \qquad j \in \{1,2\}. \tag{68}$$

Each asset contributes according to its return loading and the difference between intermediaries' and households' portfolio weights. For example, if $\sigma_{R_2}^{(1)} < 0$ and $\omega_A^{(2)} > \omega_B^{(2)}$, sector 2 makes a negative contribution to $b_{w,1}$, cushioning the fall in intermediary wealth share after an adverse sector 1 shock. A positive contribution reinforces that fall. The portfolio holdings underlying these contributions are shown in Figure 7 and discussed below.

Figure 6 shows the resulting net exposures. Both diffusion coefficients are positive in the interior of the state space under the reported calibration: an adverse shock to either sector reduces intermediary wealth share on impact. Intermediary wealth share therefore remains exposed to adverse shocks, despite the hedging opportunities available in some states. At a given wealth share, exposure to the first shock generally increases with x, while exposure to the second decreases. The figure also shows non-monotonicity in w, reflecting both endogenous

portfolio and return exposures and the factor $w(1-w)$ in (68).[24] Together, the return-loading and wealth-share figures show how capital composition changes the transmission of shocks through asset prices and intermediary wealth.

Figure 6: Diffusion of intermediary wealth share



This figure shows the full diffusion coefficients of intermediary wealth share under the unconstrained calibration as functions of x and w. The left and right panels report $b_{w,1} = w(1-w)\sigma_w^{(1)}$ and $b_{w,2} = w(1-w)\sigma_w^{(2)}$, respectively, as defined in (68). These coefficients measure the net instantaneous wealth-share response to each fundamental shock.

5.3.4 Portfolio choice, risk sharing, and risk allocation

The portfolio results show how differences in investors' risk tolerance shape their holdings of risky claims. Under the unconstrained calibration, the less risk-averse intermediaries take leveraged positions in sectoral equity and bear greater exposure per unit of wealth than households, illustrating their role as risk bearers. [25] Figure 7 reports sectoral equity holdings relative to net worth for both investor types as functions of capital composition and intermediary wealth share.

Intermediary leverage is countercyclical in w under the unconstrained calibration. When w is close to zero and sector 1 accounts for a large share of capital, intermediaries' holdings of its equity are around three to five times their net worth. At moderate wealth shares between 0.2 and 0.4, these holdings fall to around two to three times net worth. As w approaches one, intermediaries take little leverage. Households hold smaller positions relative to their wealth, as shown in the bottom panels.

[24] Because this factor approaches zero at the wealth-share boundaries, a small diffusion coefficient near a boundary does not by itself indicate weak financial amplification.

[25] This risk-bearing role is central to intermediary macro-finance models (He and Krishnamurthy, 2013; Brunnermeier and Sannikov, 2014).

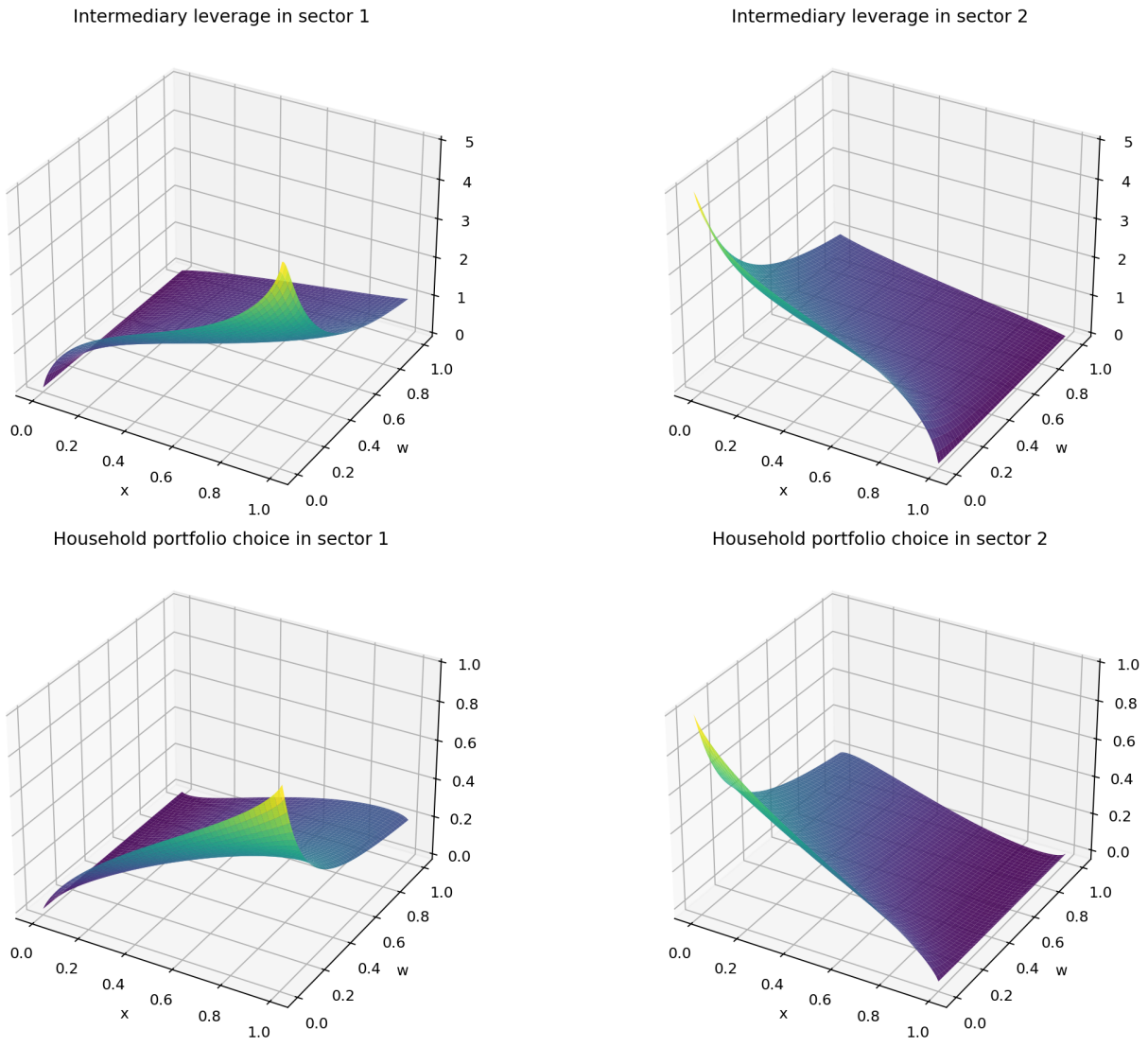
The two-sector structure also illustrates intermediaries' role as risk allocators. Holding w fixed, Figure 7 shows that their equity holdings relative to net worth shift toward sector 1 as its capital share x rises, and toward sector 2 as x falls. The risk carried by these positions depends on how each asset responds to the two fundamental shocks, as shown by the signed return loadings in Figure 4. Where one asset loads negatively on a shock, its holdings can partly offset exposure from the other asset. Together with household portfolio choices, these holdings determine how exposure to the two fundamental shocks is shared across investors.

Proposition 6 connects these exposures to investors' risk-sharing incentives. The unconstrained portfolio condition (37) combines myopic demand for risk with demand to hedge changes in investment opportunities. Higher market prices of risk at low w raise the myopic component of desired exposure, while hedging demand also depends on how investment opportunities respond to shocks. As (45) shows, R_x and R_w capture differences in how investors' marginal values of wealth respond to changes in capital composition and wealth distribution, respectively. These differences shape investors' hedging demands and thereby influence how aggregate risks are shared.

Figure 8 shows that R_x is positive when x is close to zero and negative when x is close to one, whereas R_w is positive across the plotted state space. The sign reversal in R_x means that intermediaries' log marginal value of wealth responds more positively to an increase in x than households' near $x = 0$, and less positively near $x = 1$. Changes in x alter the mix of productive risks and the investment opportunities facing both investors; changes in w redistribute wealth between investors with different risk tolerance. The R_x term therefore captures an additional source of differences in hedging incentives that arises when productive composition can change.

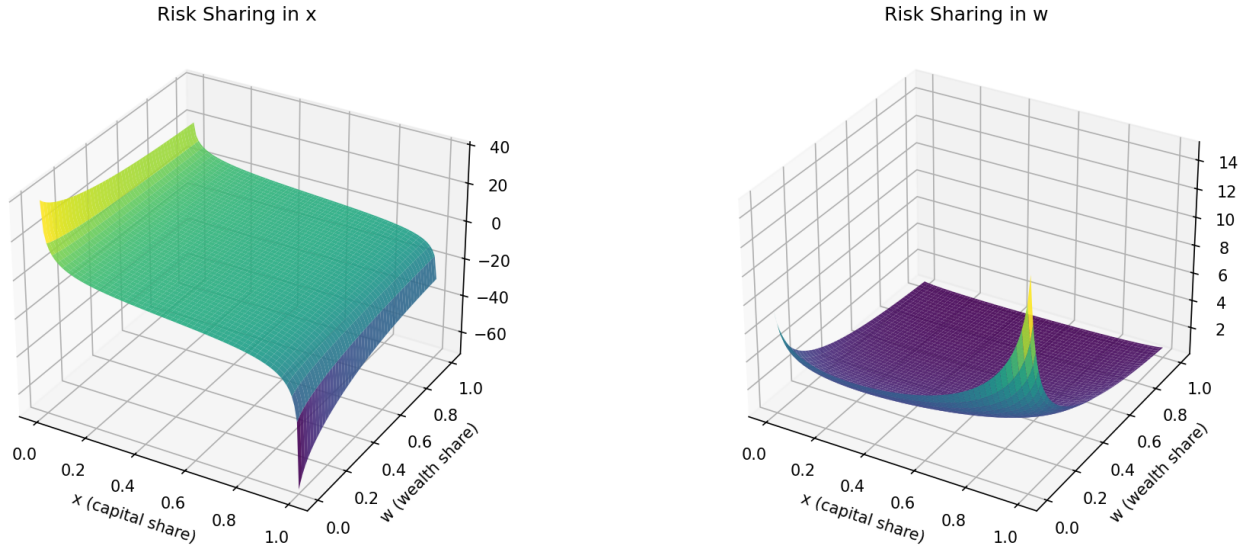
The portfolio revaluation terms Q_x and Q_w in Proposition 6 describe the valuation side of this interaction. They average the proportional sensitivities of q_1 and q_2 to x and w, respectively, using each sector's share of total equity value as its weight. Through (31), these valuation responses modify the return exposures of the aggregate equity portfolio that investors must hold. Figure 9 shows that Q_x is negative near $x = 0$ and positive near $x = 1$, with large absolute values near the boundaries. Thus, the valuation component of aggregate risk exposure depends on which sector accounts for most productive capital. The figure also shows that Q_w is positive and generally larger when w is low. Holding capital composition fixed, the positive Q_w implies that weaker intermediary capitalization lowers the market value per unit of aggregate capital, consistent with the *balance-sheet* channel, even though individual sectoral valuations can respond differently.

Figure 7: Portfolio choices



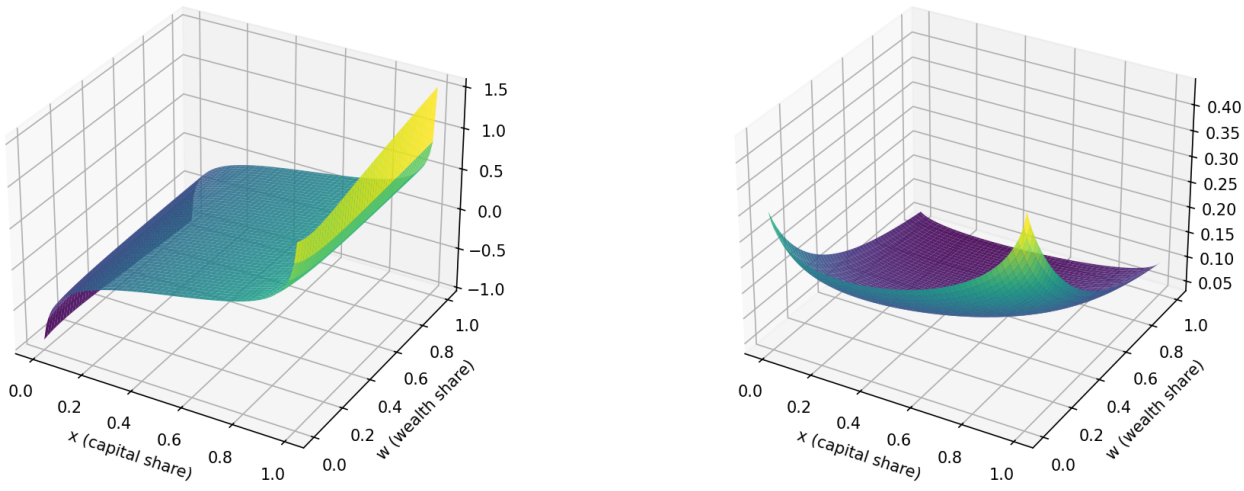
This figure shows sectoral equity holdings relative to investor net worth under the unconstrained calibration as functions of capital composition x and intermediary wealth share w. The top and bottom rows correspond to intermediaries and households, respectively; the left and right columns correspond to sectors 1 and 2. Each panel reports the portfolio weight $\omega_i^{(n)}$, measured as the value of the investor's holdings of sector-n equity divided by the investor's wealth.

Figure 8: Risk sharing R_x and R_w



This figure shows the risk-sharing terms from Proposition 6 as functions of state variables x (capital share of sector 1) and w (wealth share of intermediaries) under the unconstrained calibration. The left panel presents the risk-sharing term R_x, and the right panel presents the risk-sharing term R_w.

Figure 9: Portfolio revaluation Q_x and Q_w



This figure shows the portfolio revaluation terms from Proposition 6 as functions of state variables x (capital share of sector 1) and w (wealth share of intermediaries) under the unconstrained calibration. The left panel presents Q_x, and the right panel presents Q_w.

Together, these results connect intermediaries' allocation of risk-bearing capacity to productive capital allocation. Equilibrium portfolios and risk prices reflect both capital composition and wealth distribution, and the resulting valuations guide sectoral investment. Investment and portfolio returns subsequently change x and w, feeding back into the risks to be shared and investors' capacity to bear them.

5.4 Capital misallocation

This section examines how constraints on intermediary risk bearing affect sectoral capital allocation. Section 5.3 shows how intermediary wealth and productive composition jointly determine risk sharing and sectoral valuations in the unconstrained economy. With financial constraints, these same forces also determine the scarcity of intermediary risk-bearing capacity and the constraint's effects across sectoral claims. I compare portfolio choices, risk prices, and valuations with the unconstrained equilibrium at the same state, and then examine how the resulting investment differences change the evolution of capital composition.

To study the impact of financial frictions on capital allocation, I solve the model with the margin constraint in (13), $\|\boldsymbol{\sigma}_i\| \leq \zeta$, which bounds investors' total exposure to the fundamental shocks. All investors face this constraint, but it binds only for intermediaries in the reported numerical analysis. Their greater risk tolerance leads them to take leveraged equity positions. As intermediaries' balance sheets fall, less intermediary wealth is available to absorb aggregate risk, while their desired exposure per unit of wealth rises in the unconstrained benchmark (Figure 7). The risk limit consequently becomes restrictive in states with weak intermediary capitalization.

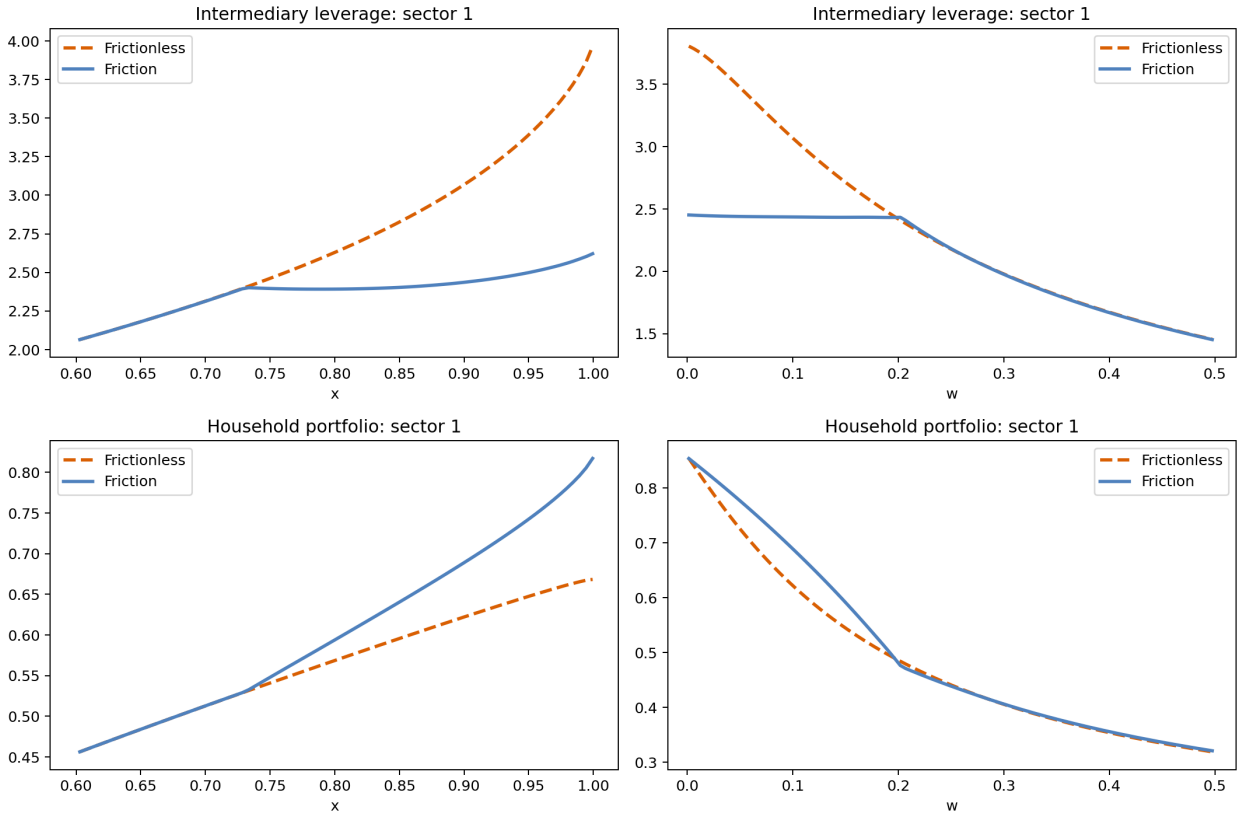
I evaluate investment distortions relative to the unconstrained equilibrium at the same state (x, w). Let $q_n^u(x, w)$ and $q_n^c(x, w)$ denote sector n's price of capital in the unconstrained and constrained equilibria, respectively. The valuation difference, $\Delta q_n(x, w) \equiv q_n^c(x, w) - q_n^u(x, w)$, measures how the presence of the financial constraint changes the value of installed capital. Since optimal investment is increasing in Q, its sign also identifies the direction of the sectoral investment distortion. The comparison identifies how introducing the portfolio constraint changes that allocation.

When the constraint binds, intermediaries' total risk exposure reaches ζ, limiting their ability to absorb risk. Their adjustment across sectoral claims depends on how each position contributes to portfolio risk (Proposition 5).Capital composition x affects how they allocate limited risk-bearing capacity across assets. The following comparisons focus on states in which sector 1, the riskier and more productive sector, accounts for a large share of capital. They trace how the constraint changes the allocation of sectoral equity across investors, the

compensation required to bear risk, and the valuations guiding investment.

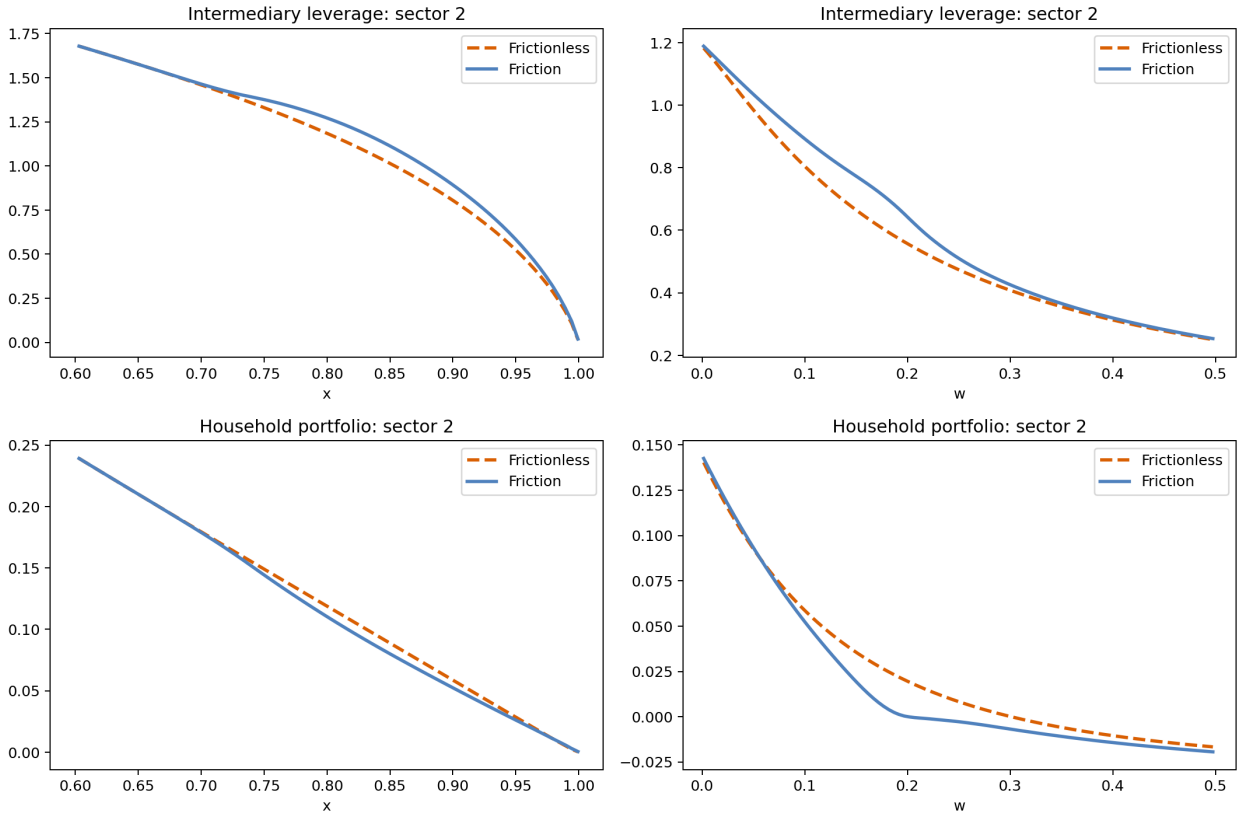
The top-right panel of Figure 10 compares intermediary portfolio weights at $x = 0.9$. When w is low and the constraint binds, intermediaries allocate a smaller fraction of their wealth to sector 1's equity than in the unconstrained equilibrium at the same state. The second risky asset provides an additional margin of adjustment: the top-right panel of Figure 11 shows a larger portfolio weight on sector 2's equity. The left panels of these figures compare portfolio choices across capital compositions, holding $w = 0.1$. At high values of x, intermediaries again allocate less wealth to sector 1's equity and more to sector 2's than in the unconstrained benchmark. The shortfall in sector 1's portfolio weight relative to the benchmark widens as x approaches one. The constraint limits intermediaries' total risk bearing, while capital composition and intermediary capitalization shape how they allocate the available risk-bearing capacity across assets.

Figure 10: Portfolio choice in sector 1



This figure compares portfolio weights in sector 1 under the total-risk constraint $\|\boldsymbol{\sigma}_i\| \leq \zeta$ (solid blue lines) and in the unconstrained benchmark (dashed orange lines), evaluated at the same state (x, w). The top panels show intermediaries' portfolio weights and the bottom panels show households' portfolio weights. The left panels vary x, the capital share of sector 1, holding $w = 0.1$; the right panels vary w, intermediaries' wealth share, holding $x = 0.9$.

Figure 11: Portfolio choice in sector 2



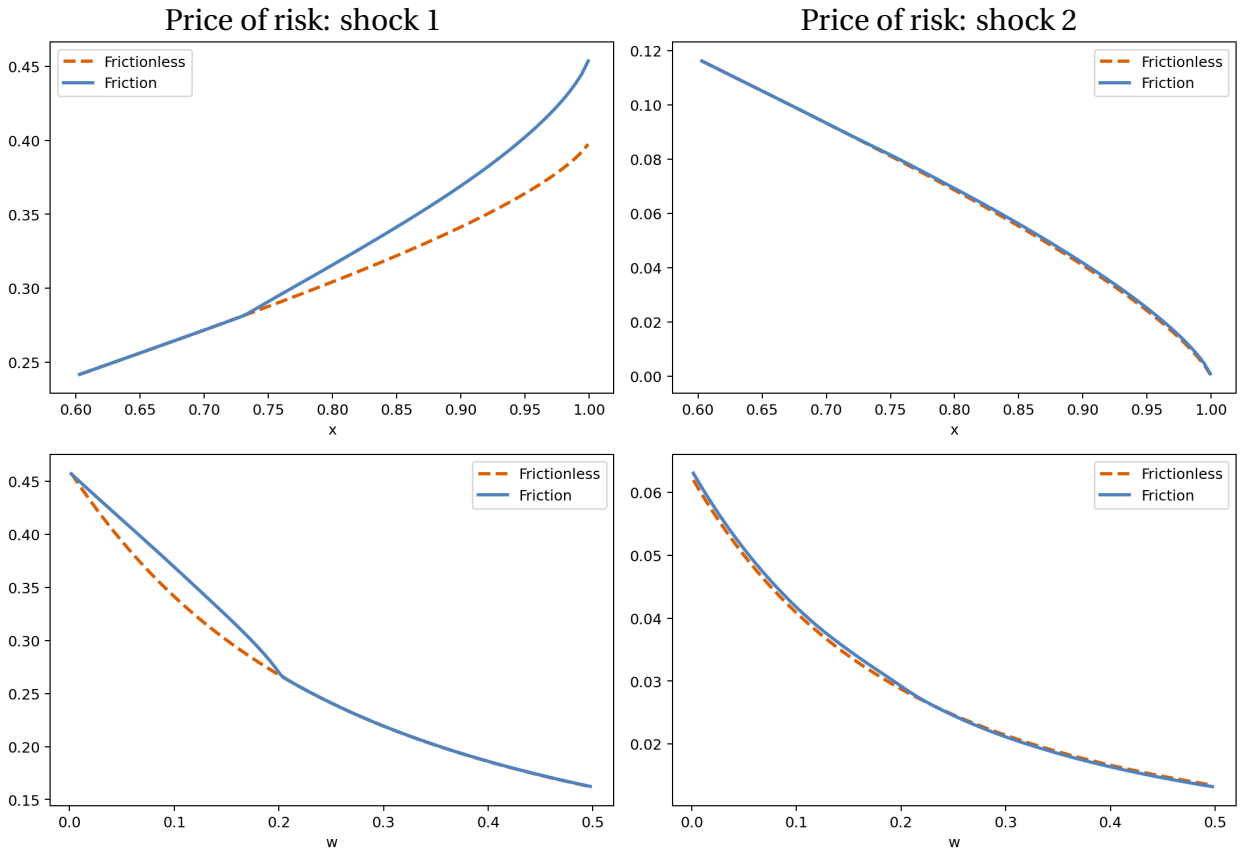
This figure compares portfolio weights in sector 2 under the total-risk constraint $\|\boldsymbol{\sigma}_i\| \leq \zeta$ (solid blue lines) and in the unconstrained benchmark (dashed orange lines), evaluated at the same state (x, w). The top panels show intermediaries' portfolio weights and the bottom panels show households' portfolio weights. The left panels vary x, the capital share of sector 1, holding $w = 0.1$; the right panels vary w, intermediaries' wealth share, holding $x = 0.9$.

The bottom panels of these figures show corresponding changes in portfolio weights of households. In these states, households allocate a larger fraction of their wealth to sector 1's equity and a smaller fraction to sector 2's than in the unconstrained benchmark. The constraint therefore changes which investors hold the sectoral claims. Risk sharing shifts toward more risk-averse households, who bear more of sector 1's risk. The price of that risk, η_1, is correspondingly higher than in the unconstrained benchmark (left panels of Figure 12).[26] The binding constraint creates a wedge between the constrained and unconstrained equilibrium prices of risk, while the equilibrium asset-pricing conditions (51) remain satisfied: each sector's equity premium reflects its return exposures to both fundamental shocks and their market prices of risk.

Distortions in risk pricing due to the binding constraint affect firms' valuations and in-

[26] The change in η_2 is comparatively small in these states.

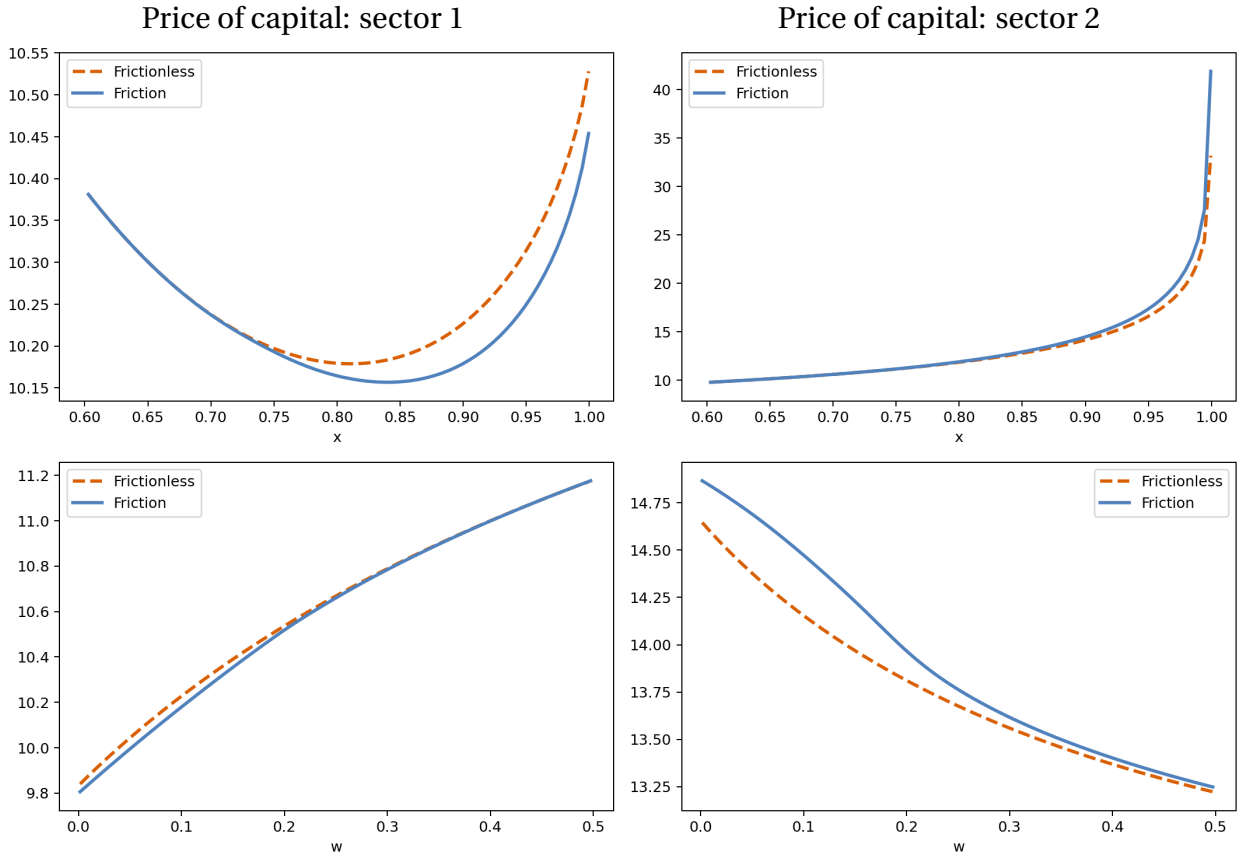
Figure 12: Financial constraint and price of risk



This figure compares the market prices of fundamental risk under the total-risk constraint $\|\boldsymbol{\sigma}_i\| \leq \zeta$ (solid blue lines) and in the unconstrained benchmark (dashed orange lines), evaluated at the same state (x, w). The left panels show η_1, the price of risk for shock B_1, and the right panels show η_2, the price of risk for shock B_2. The top panels vary x, holding $w = 0.1$; the bottom panels vary w, holding $x = 0.9$.

vestment. In the bottom panels of Figure 13, sector 1's price of capital is lower than in the unconstrained equilibrium at the same state, while sector 2's is higher. The top panels show the same pattern at high values of x, holding $w = 0.1$. Firms therefore choose a lower investment rate in sector 1 and a higher investment rate in sector 2 than in the benchmark, although their choices remain optimal given equilibrium valuations. The investment distortion arises through the allocation and pricing of risk in financial markets; firms themselves face no additional financing constraint. The distortions are concentrated in states with weak intermediary capitalization, connecting the results to the literature on cyclical capital reallocation (Eisfeldt and Rampini, 2006). My model studies a complementary margin: impaired financial risk sharing distorts sectoral investment and subsequent capital accumulation.

Figure 13: Financial constraint and sectoral valuations



This figure compares sectoral prices of capital (Tobin's Q) under the total-risk constraint $\|\boldsymbol{\sigma}_i\| \leq \zeta$ (solid blue lines) and in the unconstrained benchmark (dashed orange lines), evaluated at the same state (x, w). The left panels show q_1 and the right panels show q_2. The top panels vary x, holding $w = 0.1$; the bottom panels vary w, holding $x = 0.9$. Differences in Q have the same sign as differences in sectoral investment rates.

These investment distortions change the subsequent evolution of capital composition. Productive capital is sector specific, so differences in sectoral investment alter capital com-

position through differential accumulation. Let $\Delta g_n(x,w) \equiv g_n^c(x,w) - g_n^u(x,w)$. Since the fundamental shock exposures are unchanged across the two economies, Proposition 2 implies

$$\mu_x^c(x,w) - \mu_x^u(x,w) = x(1-x)\big[\Delta g_1(x,w) - \Delta g_2(x,w)\big]. \tag{69}$$

When sector 1 is dominant and the constraint binds, lower investment and capital growth in sector 1 and higher investment and growth in sector 2 reduce the drift of sector 1's capital share relative to the unconstrained benchmark. This is a local change in expected capital composition; the diffusion coefficients of x in (18) are identical across the two equilibria at the same state. These results show that a common portfolio risk constraint can affect relative sectoral valuations and investment even without sector-specific restrictions. Through differential capital accumulation, these investment responses change the productive composition underlying subsequent risk sharing and pricing.

6 Conclusion

This paper identifies a risk-sharing channel of capital misallocation in a production economy with heterogeneous investors. Intermediaries allocate risk-bearing capacity across sectoral assets, influencing how risks are shared and priced. When the riskier, more productive sector is dominant and intermediaries are constrained, households bear more of that sector's risk. Relative to the unconstrained equilibrium at the same state, the equilibrium price of its fundamental risk is higher. The resulting equilibrium adjustment features a lower valuation and investment in the riskier sector, while those of the safer sector are higher. Financial constraints thus affect capital allocation through the compensation required to bear sectoral risks, even though firms respond optimally to equilibrium prices.

These investment distortions change the evolution of capital composition through differential sectoral accumulation. In the states described above, they make the drift of the riskier sector's capital share more negative than in the unconstrained benchmark. Capital composition, in turn, shapes the risks investors must share, influencing subsequent portfolio choices, market prices of risk, and valuations. The model thus characterizes a two-way interaction between financial risk sharing and productive capital accumulation, connecting intermediaries' portfolio allocation to the evolution of the productive structure.

These findings have implications for policies that affect intermediaries' risk-bearing capacity. A constraint on total portfolio risk can change relative costs of capital and investment across sectors. It also links portfolio choices across assets, providing a channel through which shocks to one sector affect valuations and investment in others. Evaluating such policies there-

fore calls for attention to both intermediary capitalization and the sectoral allocation of investment, including how capital accumulation changes the economy's subsequent risk exposure. A welfare analysis of alternative risk constraints is a natural next step.

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Appendices

A Proofs

Proof of the propositions and lemmas in the main text. To simplify notation, time subscripts are dropped when it is convenient, though it should be understood that they apply.

A.1 Proposition 1

Proof. Let $W_{i,t}(s)$ denote the time-t wealth of an individual investor of type-i born at time-s. The law of motion of $W_{i,t}(s)$ is given by the budget constraint (12)

$$\frac{dW_{i,t}(s)}{W_{i,t}(s)} = \mu_{i,t}(s)dt + \boldsymbol{\sigma}_{i,t}(s)'d\boldsymbol{B}_t. \tag{70}$$

As investors of the same type face the same optimization problem, regardless of their cohort/age, we have $\mu_{i,t}(s) = \mu_{i,t}$ and $\boldsymbol{\sigma}_{i,t}(s) = \boldsymbol{\sigma}_{i,t}$, i.e., the wealth dynamics do not depend on s (when they were born).

Define the time-t average wealth of all investors of type i, including investors from different cohorts/ages s, as $W_{i,t} = \int_{-\infty}^{t} \kappa e^{-\kappa(t-s)} W_{i,t}(s)ds$. Applying Itô's lemma, the law of motion of $W_{i,t}$ is given by

$$dW_{i,t} = \frac{\partial W_{i,t}}{\partial t}dt + \frac{\partial W_{i,t}}{\partial W_{i,t}(s)}dW_{i,t}(s) \tag{71}$$

Applying Leibniz rule to the first term of (71), we have

$$\begin{aligned}\frac{\partial W_{i,t}}{\partial t} &= \kappa W_{i,t}(t) - \kappa \int_{-\infty}^{t} \kappa e^{-\kappa(t-s)} W_{i,t}(s)ds \\ &= \kappa W_{i,t}(t) - \kappa W_{i,t},\end{aligned} \tag{72}$$

using the definition of $W_{i,t}$. The last term of (71) can be written as

$$\begin{aligned}\frac{\partial W_{i,t}}{\partial W_{i,t}(s)}dW_{i,t}(s) &= \int_{-\infty}^{t} \kappa e^{-\kappa(t-s)} dW_{i,t}(s)ds \\ &= \int_{-\infty}^{t} \kappa e^{-\kappa(t-s)} W_{i,t}(s)ds(\mu_{i,t}dt + \boldsymbol{\sigma}_{i,t}'d\boldsymbol{B}_t) \\ &= W_{i,t}(\mu_{i,t}dt + \boldsymbol{\sigma}_{i,t}'d\boldsymbol{B}_t),\end{aligned} \tag{73}$$

using (70) and the definition of $W_{i,t}$ again.

Using the above two equations in (71), the law of motion of the average wealth of type-i

investors can be written as

$$dW_{i,t} = \kappa W_{i,t}(t)dt - \kappa W_{i,t}dt + W_{i,t}(\mu_{i,t}dt + \boldsymbol{\sigma}'_{i,t}d\boldsymbol{B}_t). \tag{74}$$

The first term in (74) corresponds to the wealth of newborn agents at time-t. Newborns of type-i inherit the wealth from their parents, which can be of type-A with probability $\bar{u}$ or type-B with probability $1-\bar{u}$. Hence, $W_{i,t}(t) = \bar{u}W_{A,t} + (1-\bar{u})W_{B,t}$.

Therefore, the law of motion of $W_{i,t}$ can be written as

$$dW_{i,t} = \kappa[\bar{u}W_{A,t} + (1-\bar{u})W_{B,t} - W_{i,t}]dt + W_{i,t}(\mu_{i,t}dt + \boldsymbol{\sigma}'_{i,t}d\boldsymbol{B}_t). \tag{75}$$

For $i = A$, the law of motion of $W_{A,t}$ is given by

$$\frac{dW_{A,t}}{W_{A,t}} = \kappa\left(\frac{\bar{u}}{w_t} - 1\right)dt + (\mu_{A,t}dt + \boldsymbol{\sigma}'_{A,t}d\boldsymbol{B}_t). \tag{76}$$

For $i = B$, the law of motion of $W_{B,t}$ is given by

$$\frac{dW_{B,t}}{W_{B,t}} = \kappa\left(\frac{1-\bar{u}}{1-w_t} - 1\right)dt + (\mu_{B,t}dt + \boldsymbol{\sigma}'_{B,t}d\boldsymbol{B}_t). \tag{77}$$

Denote $S_t = S_{1,t} + S_{2,t}$, and use the market clearing condition

$$\bar{u}W_{A,t} + (1-\bar{u})W_{B,t} = S_{1,t} + S_{2,t} = S_t \tag{78}$$

Apply Itô's lemma to the definition of w_t:

$$w_t = \frac{\bar{u}W_{A,t}}{\bar{u}W_{A,t} + (1-\bar{u})W_{B,t}} = \frac{\bar{u}W_{A,t}}{S_t}, \tag{79}$$

we have

$$dw_t = w_t\left[\frac{dW_{A,t}}{W_{A,t}} - \frac{dS_t}{S_t} + \left(\frac{dS_t}{S_t}\right)^2 - \frac{dS_t}{S_t}\frac{dW_{A,t}}{W_{A,t}}\right], \tag{80}$$

in which we can use

$$\frac{dS_t}{S_t} = w_t\frac{dW_{A,t}}{W_{A,t}} + (1-w_t)\frac{dW_{B,t}}{W_{B,t}}. \tag{81}$$

from the definition of S_t. The terms inside the brackets of (80) can be written as

$$\frac{dW_{A,t}}{W_{A,t}} - \frac{dS_t}{S_t} = (1-w_t)\left(\frac{dW_{A,t}}{W_{A,t}} - \frac{dW_{B,t}}{W_{B,t}}\right)$$
$$\left(\frac{dS_t}{S_t}\right)^2 - \frac{dS_t}{S_t}\frac{dW_{A,t}}{W_{A,t}} = -(1-w_t)\left(\frac{dW_{A,t}}{W_{A,t}} - \frac{dW_{B,t}}{W_{B,t}}\right)\left[w_t\frac{dW_{A,t}}{W_{A,t}} + (1-w_t)\frac{dW_{B,t}}{W_{B,t}}\right]$$

Substituting the above two equations into (80) and using (76) and (77), we have

$$\begin{aligned} dw_t = {} & \kappa[\bar{u} - w_t]dt \\ & + (1-w_t)w_t\left[(\boldsymbol{\sigma}'_{A,t} - \boldsymbol{\sigma}'_{B,t})\boldsymbol{\eta}_t + c_{B,t} - c_{A,t}\right]dt \\ & - (1-w_t)w_t\left[(\boldsymbol{\sigma}'_{A,t} - \boldsymbol{\sigma}'_{B,t})\boldsymbol{\Sigma}'_{R,t}\begin{pmatrix} z_t \\ 1-z_t \end{pmatrix}\right]dt \\ & + (1-w_t)w_t(\boldsymbol{\sigma}'_{A,t} - \boldsymbol{\sigma}'_{B,t})d\boldsymbol{B}_t, \end{aligned} \tag{82}$$

The above expression gives the law of motion for w, as shown in the main text. ■

A.2 Proposition 2

Proof. Define $K = K_1 + K_2$, $x = \frac{K_1}{K_1+K_2}$. Use the law of motion of K_1 and K_2

$$\frac{dK_1}{K_1} = g_1 dt + \sigma_1 dB_1,$$
$$\frac{dK_2}{K_2} = g_2 dt + \sigma_2 dB_2.$$

Applying Itô's lemma to K and substituting the definition of K and x, we have

$$\begin{aligned} d(K_1+K_2) &= (g_1K_1 + g_2K_2)dt + \sigma_1 K_1 dB_1 + \sigma_2 K_2 dB_2, \\ dK &= (g_1 xK + g_2(1-x)K)dt + \sigma_1 xK\, dB_1 + \sigma_2 K(1-x)dB_2. \end{aligned} \tag{83}$$

Applying Itô's lemma to x, we have

$$\begin{aligned} dx &= \frac{1}{K}dK_1 - \frac{K_1}{K^2}dK + \frac{K_1}{K^3}(dK)^2 - \frac{1}{K^2}dK_1 dK \\ &= x(g_1 dt + \sigma_1 dB_1) - x[(g_1 x + g_2(1-x))dt + \sigma_1 x dB_1 + \sigma_2(1-x)dB_2] \\ &\quad + x(\sigma_1^2 x^2 + \sigma_2^2(1-x)^2)dt - x\sigma_1^2 x dt \\ &= x(1-x)\left[g_1 - g_2 - \sigma_1^2 x + \sigma_2^2(1-x)\right]dt + \sigma_1 x(1-x)dB_1 - \sigma_2 x(1-x)dB_2. \end{aligned} \tag{84}$$

The above expression gives the law of motion for x, as shown in the main text. ■

A.3 Proposition 3

Proof. To simplify notation, time and sector n subscripts are dropped, as the proposition applies to both sectors at any time.

HJB equation for the firm's problem is given by

$$0 = \max_{\iota} \left\{ D\,dt + \mathbb{E}\left[\frac{d(\Lambda S)}{\Lambda} \right] \right\}, \tag{85}$$

where D is the dividend paid by the firm, Λ is the SDF.

We have the following law of motions

$$\begin{aligned}
\frac{dK}{K} &= g\,dt + \sigma_1 dB_1 + \sigma_2 dB_2, \\
\frac{dq}{q} &= \mu_q dt + \sigma_{q,1} dB_1 + \sigma_{q,2} dB_2, \\
\frac{d\Lambda}{\Lambda} &= -r\,dt - \eta_1 dB_1 - \eta_2 dB_2.
\end{aligned}$$

Applying Itô's lemma to $S = qK$, we have

$$\begin{aligned}
dS &= q\,dK + K\,dq + dq\,dK, \\
\frac{dS}{S} &= \left(g + \mu_q + \sigma_1\sigma_{q,1} + \sigma_2\sigma_{q,2}\right) dt + \left(\sigma_1 + \sigma_{q,1}\right) dB_1 + \left(\sigma_2 + \sigma_{q,2}\right) dB_2, \\
&= \mu_S dt + \sigma_{S,1} dB_1 + \sigma_{S,2} dB_2.
\end{aligned}$$

Applying Itô's lemma to ΛS, we have

$$\frac{d\Lambda S}{\Lambda S} = \left(\mu_S - r - \eta_1\sigma_{S,1} - \eta_2\sigma_{S,2}\right) dt + (\sigma_{S,1} - \eta_1) dB_1 + (\sigma_{S,2} - \eta_2) dB_2. \tag{86}$$

Equating the drift from (86) to D, the HJB equation for the firm's problem is thus given by

$$0 = \max_{\iota} \left\{ \frac{A - \iota}{q} + g + \mu_q + \boldsymbol{\sigma}_q' \boldsymbol{\sigma} - r - \boldsymbol{\eta}'(\boldsymbol{\sigma}_q + \boldsymbol{\sigma}) \right\} \tag{87}$$

where the bold symbols denote vectors for simplicity.

The optimal investment rate is given by the first order condition

$$g'(\iota) = 1/q. \tag{88}$$

Note that using Itô's lemma, the drift and diffusion μ_q, $\sigma_{q,1}$ and $\sigma_{q,2}$ can be expressed as follows

$$\begin{aligned}\mu_q = &\frac{q_x}{q}\mu_x + \frac{q_w}{q}\left[\kappa(\bar{\mu} - w) + w(1-w)\mu_w\right] \\ &+ \frac{1}{2}\frac{q_{xx}}{q}(\sigma_{x,1}^2 + \sigma_{x,2}^2) + \frac{1}{2}\frac{q_{ww}}{q}w^2(1-w)^2(\sigma_{w,1}^2 + \sigma_{w,2}^2) \\ &+ \frac{q_{xw}}{q}w(1-w)(\sigma_{x,1}\sigma_{w,1} + \sigma_{x,2}\sigma_{w,2}), \qquad (89)\\ \sigma_{q,1} = &\frac{q_x}{q}\sigma_{x,1} + \frac{q_w}{q}w(1-w)\sigma_{w,1}, \qquad (90)\\ \sigma_{q,2} = &\frac{q_x}{q}\sigma_{x,2} + \frac{q_w}{q}w(1-w)\sigma_{w,2}. \qquad (91)\end{aligned}$$

where $q_x = \partial q/\partial x$, $q_w = \partial q/\partial w$, $q_{xx} = \partial^2 q/\partial x^2$, $q_{ww} = \partial^2 q/\partial w^2$, $q_{xw} = \partial^2 q/\partial x \partial w$. $\mu_x, \mu_w, \sigma_{x,1}$, $\sigma_{x,2}, \sigma_{w,1}, \sigma_{w,2}$ are from the law of motion of state variables x and w, (18) and (17), respectively. ■

A.4 Proposition 4

Proof. To simplify notation, time and investor type i subscripts are dropped, as the proposition applies to investors of both types at any time.

HJB equation for the investor's problem is given by

$$0 = \max_{\substack{c\geq 0 \\ \|\boldsymbol{\sigma}\|\leq\zeta}} \{f(cW, V)dt + \mathbb{E}[dV]\}, \qquad (92)$$

where f is the aggregator of Epstein-Zin preferences.

We have the following law of motions

$$\begin{aligned}\frac{dW}{W} &= \mu_W dt + \sigma_{W,1}dB_1 + \sigma_{W,2}dB_2,\\ \frac{d\xi}{\xi} &= \mu_\xi dt + \sigma_{\xi,1}dB_1 + \sigma_{\xi,2}dB_2.\end{aligned}$$

Applying Itô's lemma to

$$V = \frac{W^{1-\gamma}}{1-\gamma}\xi^{\frac{1-\gamma}{1-\psi}},$$

we have

$$dV = \frac{\partial V}{\partial W}dW + \frac{\partial V}{\partial \xi}d\xi + \frac{1}{2}\frac{\partial^2 V}{\partial W^2}(dW)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \xi^2}(d\xi)^2 + \frac{\partial^2 V}{\partial W \partial \xi}dWd\xi,$$

in which

$$\begin{aligned}\frac{\mathbb{E}[dV]}{dt} &= (1-\gamma)V\mu_W + \frac{1-\gamma}{1-\psi}V\mu_\xi + \frac{1}{2}(1-\gamma)(-\gamma)V(\sigma_{W,1}^2+\sigma_{W,2}^2)\\ &+ \frac{1}{2}\frac{1-\gamma}{1-\psi}\left(\frac{1-\gamma}{1-\psi}-1\right)V(\sigma_{\xi,1}^2+\sigma_{\xi,2}^2)\\ &+ (1-\gamma)\frac{1-\gamma}{1-\psi}V(\sigma_{W,1}\sigma_{\xi,1}+\sigma_{W,2}\sigma_{\xi,2}).\end{aligned} \tag{93}$$

Meanwhile,

$$f(cW,V) = \frac{1-\gamma}{1-1/\psi}V\left[\left(\frac{cW}{((1-\gamma)V)^{\frac{1}{1-\gamma}}}\right)^{1-1/\psi}-\rho\right] \tag{94}$$

Equate the sum of (93) and (94) to zero and remove the term $(1-\gamma)V$, we have

$$\begin{aligned}0 = \max_{\substack{c\geq 0\\ \|\boldsymbol{\sigma}\|\leq\zeta}}\Bigg\{&\frac{1}{1-1/\psi}\left(c^{1-1/\psi}\xi^{1/\psi}-\rho\right)+[\boldsymbol{\sigma}'\boldsymbol{\eta}_t + r_t - c]\\ &+\frac{1}{1-\psi}\mu_\xi - \frac{\gamma}{2}\boldsymbol{\sigma}'\boldsymbol{\sigma}\\ &+\frac{1}{2}\frac{1}{1-\psi}\left(\frac{1-\gamma}{1-\psi}-1\right)\boldsymbol{\sigma}_\xi'\boldsymbol{\sigma}_\xi\\ &+\frac{1-\gamma}{1-\psi}\boldsymbol{\sigma}'\boldsymbol{\sigma}_\xi\Bigg\},\end{aligned} \tag{95}$$

where matrix notation is used for simplicity. Holding prices and continuation values fixed, the consumption first-order condition and the risk-exposure first-order condition with the portfolio constraint temporarily ignored give

$$c = \xi, \tag{96}$$

$$\boldsymbol{\sigma}^* = \frac{1}{\gamma}\boldsymbol{\eta} + \frac{1}{\gamma}\frac{1-\gamma}{1-\psi}\boldsymbol{\sigma}_\xi. \tag{97}$$

The constrained optimal exposure is given by (38).

Note that using Itô's lemma, the drift and diffusion μ_ξ, $\sigma_{\xi,1}$ and $\sigma_{\xi,2}$ can be expressed as

follows

$$\mu_\xi = \frac{\xi_x}{\xi}\mu_x + \frac{\xi_w}{\xi}\left[\kappa(\bar{\mu} - w) + w(1-w)\mu_w\right] + \frac{1}{2}\frac{\xi_{xx}}{\xi}(\sigma_{x,1}^2 + \sigma_{x,2}^2) + \frac{1}{2}\frac{\xi_{ww}}{\xi}w^2(1-w)^2(\sigma_{w,1}^2 + \sigma_{w,2}^2) + \frac{\xi_{xw}}{\xi}w(1-w)(\sigma_{x,1}\sigma_{w,1} + \sigma_{x,2}\sigma_{w,2}), \tag{98}$$

$$\sigma_{\xi,1} = \frac{\xi_x}{\xi}\sigma_{x,1} + \frac{\xi_w}{\xi}w(1-w)\sigma_{w,1}, \tag{99}$$

$$\sigma_{\xi,2} = \frac{\xi_x}{\xi}\sigma_{x,2} + \frac{\xi_w}{\xi}w(1-w)\sigma_{w,2}. \tag{100}$$

where $\xi_x = \partial\xi/\partial x$, $\xi_w = \partial\xi/\partial w$, $\xi_{xx} = \partial^2\xi/\partial x^2$, $\xi_{ww} = \partial^2\xi/\partial w^2$, $\xi_{xw} = \partial^2\xi/\partial x\partial w$. μ_x, μ_w, $\sigma_{x,1}$, $\sigma_{x,2}$, $\sigma_{w,1}$, $\sigma_{w,2}$ are from the law of motion of state variables x and w, (18) and (17), respectively. ■

A.5 Proposition 5

Proof. Fix an investor type $i \in \{A, B\}$, suppress time subscripts, and define $\mathbf{p}_i \equiv \boldsymbol{\eta} + \frac{1-\gamma_i}{1-\psi}\boldsymbol{\sigma}_{\xi_i}$. Holding prices, continuation values, and aggregate state dynamics fixed, the terms in (35) that depend on the investor's exposure give the Lagrangian

$$\mathscr{L} = \boldsymbol{\sigma}_i'\mathbf{p}_i - \frac{\gamma_i}{2}\|\boldsymbol{\sigma}_i\|^2 + \nu_i\left(\zeta - \|\boldsymbol{\sigma}_i\|\right).$$

The objective is strictly concave and the feasible set is a Euclidean ball with nonempty interior, so the optimality conditions are necessary and sufficient. At a nonzero optimal exposure, differentiating the Lagrangian with respect to $\boldsymbol{\sigma}_i$ gives the first equation below; the remaining conditions impose feasibility and complementary slackness:

$$\mathbf{p}_i - \gamma_i\boldsymbol{\sigma}_i = \nu_i\frac{\boldsymbol{\sigma}_i}{\|\boldsymbol{\sigma}_i\|},$$
$$\|\boldsymbol{\sigma}_i\| \le \zeta, \qquad \nu_i \ge 0, \qquad \nu_i\left(\zeta - \|\boldsymbol{\sigma}_i\|\right) = 0.$$

If the constraint is slack, $\nu_i = 0$ and $\boldsymbol{\sigma}_i = \mathbf{p}_i/\gamma_i$. If $\|\boldsymbol{\sigma}_i\| = \zeta$, the first-order condition for risk exposure implies $\|\mathbf{p}_i\| = \gamma_i\zeta + \nu_i$. Since (37) gives $\mathbf{p}_i = \gamma_i\boldsymbol{\sigma}_i^*$, these cases yield (39). At zero optimal exposure, $\zeta > 0$ implies a slack constraint, so $\nu_i = 0$ and the constraint term vanishes.

Using $\boldsymbol{\sigma}_i = \boldsymbol{\Sigma}_R'\boldsymbol{\omega}_i$, differentiation with respect to ω_{in} gives (40). Premultiplying the first-order condition for risk exposure by $\boldsymbol{\sigma}_{R_n}'$, substituting for $\mathbf{p}_i$, and using $\mu_{R_n} - r = \boldsymbol{\sigma}_{R_n}'\boldsymbol{\eta}$ yields (41) with $\chi_{in} = \nu_i m_{in}$. Subtracting the two constraint terms gives (42). ■

A.6 Proposition 6

Proof. Denote $\sigma_A^{(1),*}$ as type A investors' optimal risk exposure to risk from sector 1, i.e., the first element of their risk exposure $\boldsymbol{\sigma}_A^*$. Using the market clearing condition for the risk exposure,

$$w\sigma_A^{(1),*} + (1-w)\sigma_B^{(1)} = z\sigma_{R,1}^{(1)} + (1-z)\sigma_{R,2}^{(1)},$$

express $\sigma_B^{(1)}$ in terms of $\sigma_A^{(1),*}$. Then plug it into the expression for investor of type B's optimal risk exposure from (37),

$$\sigma_B^{(1)} = \frac{1}{\gamma_B}\eta^{(1)} + \frac{1}{\gamma_B}\frac{1-\gamma_B}{1-\psi}\sigma_{\xi_B}^{(1)},$$

substituting for $\eta^{(1)}$ using the expression for $\sigma_A^{(1),*}$,

$$\sigma_A^{(1),*} = \frac{1}{\gamma_A}\eta^{(1)} + \frac{1}{\gamma_A}\frac{1-\gamma_A}{1-\psi}\sigma_{\xi_A}^{(1)}.$$

Then I can compute $\sigma_A^{(1),*}$ as follows:

$$\sigma_A^{(1),*} = \frac{zR_w + R_wQ_xx(1-x) - R_x(Q_w + \frac{1}{w})x(1-x) - z\frac{\gamma_B}{w(1-w)} - \frac{\gamma_B}{w(1-w)}x(1-x)Q_x}{(\gamma_B-\gamma_A)Q_w - \frac{\gamma_B}{1-w} - \frac{\gamma_A}{w} + R_w}\sigma_1, \quad (101)$$

where

$$R_x = \frac{1-\gamma_A}{1-\psi}\frac{\xi_{A,x}}{\xi_A} - \frac{1-\gamma_B}{1-\psi}\frac{\xi_{B,x}}{\xi_B}; \quad R_w = \frac{1-\gamma_A}{1-\psi}\frac{\xi_{A,w}}{\xi_A} - \frac{1-\gamma_B}{1-\psi}\frac{\xi_{B,w}}{\xi_B};$$
$$Q_x = z\frac{q_{1,x}}{q_1} + (1-z)\frac{q_{2,x}}{q_2}; \qquad Q_w = z\frac{q_{1,w}}{q_1} + (1-z)\frac{q_{2,w}}{q_2}.$$

In a similar manner, I can compute $\sigma_A^{(2),*}$, the second element of $\boldsymbol{\sigma}_A^*$, as follows:

$$\sigma_A^{(2),*} = \frac{(1-z)R_w - R_wQ_xx(1-x) + R_x(Q_w + \frac{1}{w})x(1-x) - \frac{(1-z)\gamma_B}{w(1-w)} + \frac{\gamma_B}{w(1-w)}x(1-x)Q_x}{(\gamma_B-\gamma_A)Q_w - \frac{\gamma_B}{1-w} - \frac{\gamma_A}{w} + R_w}\sigma_2. \quad (102)$$

The risk exposure of type B investors can be computed using the market clearing condition for risk exposure. ■

A.7 Proposition 7

Proof. Plugging the (frictionless) optimal risk exposure (37) for type A and B investors into the market clearing conditions of risk exposure, I derive the expression for the vector of market

price of risk

$$\begin{aligned}\boldsymbol{\eta}_t = \gamma_t \left[z_t \boldsymbol{\sigma}_{R_1,t} + (1-z_t)\boldsymbol{\sigma}_{R_2,t} \right] \\ - \gamma_t \left[w_t \left(\frac{1}{\gamma_A}\right)\left(\frac{1-\gamma_A}{1-\psi}\right)\boldsymbol{\sigma}_{\xi_A,t} + (1-w_t)\left(\frac{1}{\gamma_B}\right)\left(\frac{1-\gamma_B}{1-\psi}\right)\boldsymbol{\sigma}_{\xi_B,t} \right],\end{aligned} \tag{103}$$

where γ_t denotes aggregate risk aversion, and is given by

$$\gamma_t = \left(\frac{w_t}{\gamma_A} + \frac{1-w_t}{\gamma_B} \right)^{-1}. \tag{104}$$

■

A.8 Proposition 8

Proof. Plugging the optimal consumption-wealth ratio (36) into the HJB (35), I obtain

$$c_{i,t} = \psi\rho + \mu_{\xi_i,t} + (1-\psi)(r_t + \Phi_{i,t}), \tag{105}$$

where

$$\begin{aligned}\Phi_{i,t} = \boldsymbol{\sigma}'_{i,t}\boldsymbol{\eta}_t - \frac{\gamma_i}{2}\boldsymbol{\sigma}'_{i,t}\boldsymbol{\sigma}_{i,t} \\ + \frac{1}{2}\frac{1}{1-\psi}\left(\frac{1-\gamma_i}{1-\psi} - 1\right)\boldsymbol{\sigma}'_{\xi_i,t}\boldsymbol{\sigma}_{\xi_i,t} + \frac{1-\gamma_i}{1-\psi}\boldsymbol{\sigma}'_{i,t}\boldsymbol{\sigma}_{\xi_i,t}.\end{aligned} \tag{106}$$

Plugging the above equation into the goods market clearing condition, I obtain the expression for the interest rate

$$\psi\rho + \mu_t + (1-\psi)(r_t + \Phi_t) = z_t \frac{A_1 - \iota_1(g_{1,t})}{q_{1,t}} + (1-z_t)\frac{A_2 - \iota_2(g_{2,t})}{q_{2,t}}, \tag{107}$$

where $\mu_t = w_t\mu_{\xi_A,t} + (1-w_t)\mu_{\xi_B,t}$, and $\Phi_t = w_t\Phi_{A,t} + (1-w_t)\Phi_{B,t}$. ■

A.9 Lemma 1

Proof. The benchmark economy is essentially non-stochastic, up to scale. The equilibrium conditions are:

$$\frac{1}{1-1/\psi}(c_i-\rho)+(\sigma_i\eta+r-c_i)-\frac{\gamma_i}{2}\sigma_i^2=0,$$
$$\sigma_i=\frac{\eta}{\gamma_i},$$
$$\frac{A-\iota(g)}{q}+g-r-\eta\sigma=0,$$
$$\iota'(g)=q,$$
$$c=\frac{A-\iota(g)}{q},$$
$$\sigma_i=\sigma.$$

The market price of risk is

$$\eta^*=\gamma\sigma.$$

The interest rate is given by:

$$r^*=\rho+\psi^{-1}g^*-\left(1+\psi^{-1}\right)\frac{\gamma\sigma^2}{2}.$$

The consumption-wealth ratios, i.e., the dividend yield on aggregate wealth are

$$c_A^*=c_B^*=\rho-\left(1-\psi^{-1}\right)\left(g^*-\frac{\gamma\sigma^2}{2}\right).$$

The growth rate g^* and price of capital q^* satisfy

$$\frac{A-\iota(g^*)}{q^*}=\rho-\left(1-\psi^{-1}\right)\left(g^*-\frac{\gamma\sigma^2}{2}\right),$$
$$\iota'(g^*)=q^*.$$

■

A.10 Lemma 2

This economy is the same as the one in Eberly and Wang (2025), except that it is solved in decentralized equilibrium. Eberly and Wang (2025) solve the social planner's problem, which

is equivalent to the decentralized equilibrium in this case.

Assume $\gamma_A = \gamma_B = \gamma$ and $\kappa = 0$, i.e., there is a representative investor. In addition, assume

$$\frac{dK_{1,t}}{K_{1,t}} = g_{1,t}dt + \sigma\varepsilon dB_{1,t} + \sigma(1-\varepsilon)dB_{2,t},$$
$$\frac{dK_{2,t}}{K_{2,t}} = g_{2,t}dt + \sigma dB_{2,t}.$$

The two sectors are assumed to be symmetric for expositional simplicity: productivity A, depreciation rate δ, and adjustment cost (ϕ, θ) are assumed to be the same across two sectors.

Proof. We consider a first-order asymptotic expansion around the point $\varepsilon = 0$, i.e., for each variable z, we want to find the functions $(z_0(x), z_1(x))$ such that $z(x;\varepsilon) = z_0(x) + z_1(x)\varepsilon + O(\varepsilon^2)$, in which x, the capital share of sector 1, is state variable.

The zeroth-order coefficients are derived in the benchmark economy, meaning that the economy in Lemma 1 corresponds to the case of $\varepsilon = 0$, for which the solution can be solved explicitly. For the case where $\varepsilon > 0$, the equilibrium objects can be approximated by a Taylor expansion around $\varepsilon = 0$. For example, the price of capital in sector 1 can be approximated as

$$q_1(x;\varepsilon) = q^* + q_{1,\varepsilon}(x;0)\varepsilon + O(\varepsilon^2),$$

where q^* is the price of capital in the economy in Lemma 1. By computing the derivatives with respect to ε, $q_{1,\varepsilon}(x;0)$, we can obtain explicit expressions for the equilibrium objects when the extent of heterogeneity is small. The other equilibrium objects can be approximated similarly.

Using the above two expressions for laws of motion of capital, the diffusion of x to first shock from (18) can be written as

$$\sigma_{x,1} = x(1-x)\varepsilon\sigma,$$

and the diffusion of x to second shock can be written as

$$\sigma_{x,2} = -x(1-x)\varepsilon\sigma.$$

Since the derivative of $q(x;0)$ (zeroth-order coefficient) with respect to x is zero, the derivative of $q(x;\varepsilon)$ with respect to x, namely $q_x(x;\varepsilon)$, is first-order in ε. Similarly, the derivative of consumption-wealth ratio $\xi_i(x;\varepsilon)$ with respect to x, namely $\xi_{i,x}(x;\varepsilon)$, is also first-order in ε. Meanwhile, the diffusion terms of x are also first-order in ε as can be seen from above. Thus $\sigma_{q,1}, \sigma_{q,2}$ and $\sigma_{\xi_{i,1}}, \sigma_{\xi_{i,2}}$ are of order $O(\varepsilon^2)$ (see proofs of Proposition 3 and 4 for expressions).

Combining the above with the market prices of risks from (47), we can write the market

prices of risks as

$$\eta_1(x;\varepsilon) = \gamma\sigma z\varepsilon + O(\varepsilon^2),$$
$$\eta_2(x;\varepsilon) = \gamma(\sigma - \sigma z\varepsilon) + O(\varepsilon^2).$$

The risk exposure of investors from (37) can be written as

$$\sigma_{i,1} = \sigma z\varepsilon + O(\varepsilon^2),$$
$$\sigma_{i,2} = \sigma - \sigma z\varepsilon + O(\varepsilon^2).$$

From the optimal consumption-wealth ratio (105), we can write it as

$$c_i(x;\varepsilon) = \psi\rho + (1-\psi)(r_t + \frac{\gamma_i}{2}\sigma^2 - \gamma_i\sigma^2 z\varepsilon) + O(\varepsilon^2).$$

where it uses $\mu_{\xi_i,t} = O(\varepsilon^2)$, and $\Phi_{i,t} = \frac{\gamma_i}{2}\sigma^2 - \gamma_i\sigma^2 z\varepsilon + O(\varepsilon^2)$.

Write the drift of x from (18) as

$$\mu_x = x(1-x)\left[g_1 - g_2\right] + 2x^3\varepsilon^2\sigma^2 - 2x^2\varepsilon^2\sigma^2 - x^2\varepsilon\sigma^2 + x\varepsilon\sigma^2$$

Since derivative $c_{i,x}(x;\varepsilon)$ is first-order in ε. The drift terms of c and q are second-order in ε.

Thus the firms' HJB equations can be written as

$$\frac{A - \iota(g_n)}{q_n} + g_n + - r_t - \eta_1\sigma_{n,1} - \eta_2\sigma_{n,2} = 0.$$

where it uses $\mu_{q,n} = O(\varepsilon^2)$, and $\sigma_{q_n,1}, \sigma_{q_n,2} = O(\varepsilon^2)$.

Thus we have

$$\frac{A - \iota(g_1)}{q_1} = r_t - g_1 + \gamma\sigma^2 - \gamma\sigma^2 z\varepsilon - \gamma\sigma^2\varepsilon + O(\varepsilon^2),$$
$$\frac{A - \iota(g_2)}{q_2} = r_t - g_2 + \gamma\sigma^2 - \gamma\sigma^2 z\varepsilon + O(\varepsilon^2).$$

Use the above HJBs for investors and firms, and market clearing condition for consump-

tion goods, the interest rate can be solved as

$$\begin{aligned}
r_t &= \rho - (1-\psi^{-1})(\frac{\gamma}{2}\sigma^2 - \gamma\sigma^2 z\varepsilon) - \psi^{-1}\left[\gamma\sigma^2 - 2\gamma\sigma^2 z\varepsilon - zg_1 - (1-z)g_2\right] \\
&= \rho + \psi^{-1}g^* - (1+\psi^{-1})\frac{\gamma}{2}\sigma^2 \\
&+ (1+\psi^{-1})\gamma\sigma^2 z\varepsilon + \psi^{-1}\left[z(g_1 - g^*) + (1-z)(g_2 - g^*)\right] + O(\varepsilon^2), \\
&= r^* \\
&+ (1+\psi^{-1})\gamma\sigma^2 z\varepsilon + \psi^{-1}\left[z(g_1 - g^*) + (1-z)(g_2 - g^*)\right] + O(\varepsilon^2).
\end{aligned}$$

Following the first-order asymptotic expansion, write the price of capital as

$$\begin{aligned}
q_1 &= q^* + q_{1,1}(x)\varepsilon + O(\varepsilon^2), \\
q_2 &= q^* + q_{2,1}(x)\varepsilon + O(\varepsilon^2),
\end{aligned}$$

where q^* is the equilibrium price of capital from Lemma 1, and $q_{1,1}$ denotes the first-order coefficient of q_1 with respect to ε. Similarly for $q_{2,1}$.

From (25), $g = \frac{q-\phi}{\theta} - \delta$. Then growth rate can be written as

$$\begin{aligned}
g_1 &= g^* + \frac{q_{1,1}}{\theta}\varepsilon + O(\varepsilon^2), \\
g_2 &= g^* + \frac{q_{2,1}}{\theta}\varepsilon + O(\varepsilon^2),
\end{aligned}$$

where $g^* = \frac{q^*-\phi}{\theta} - \delta$ is the equilibrium growth rate from Lemma 1.

Similarly from (25), $\iota = \frac{q^2-\phi^2}{2\theta}$, the investment rate can be written as

$$\begin{aligned}
\iota_1 &= \frac{(q^*)^2 + 2q^* q_{1,1}\varepsilon - \phi^2}{2\theta} + O(\varepsilon^2), \\
\iota_2 &= \frac{(q^*)^2 + 2q^* q_{2,1}\varepsilon - \phi^2}{2\theta} + O(\varepsilon^2).
\end{aligned}$$

The variable z is defined in (27), which can be written as

$$z = \left(1 + \frac{q_{2,t}}{q_{1,t}}\frac{1-x_t}{x_t}\right)^{-1}.$$

Replace r_t, growth rate, price of capital, investment rate into firms' HJB equations, we can solve for the pair of $(q_{1,1}, q_{2,1})$ jointly, which gives the first-order asymptotic expansion of the

equilibrium objects. The pair of equations to be solved are the following:

$$0 = c^* q_{1,1}\varepsilon + q^*\psi^{-1}\gamma\sigma^2 z\varepsilon + \psi^{-1}q^*\left[z\frac{q_{1,1}}{\theta} + (1-z)\frac{q_{2,1}}{\theta}\right]\varepsilon - q^*\gamma\sigma^2\varepsilon,$$
$$0 = c^* q_{2,1}\varepsilon + q^*\psi^{-1}\gamma\sigma^2 z\varepsilon + \psi^{-1}q^*\left[z\frac{q_{1,1}}{\theta} + (1-z)\frac{q_{2,1}}{\theta}\right]\varepsilon.$$

where $c^* = r^* - g^* + \gamma\sigma^2$ is the consumption-wealth ratio from Lemma 1.

The solution to the above pair of equations gives

$$q_{1,1} = \frac{q^*\gamma\sigma^2(-c^* x\theta + c^*\theta\psi - q^* x + q^*)}{c^*(c^*\theta\psi + q^*)},$$
$$q_{2,1} = \frac{q^* x\gamma\theta\sigma^2(-c^*\theta - q^*)}{c^*(c^*\theta\psi + q^*)}.$$

The derivative of price of capital with respect to x, same for both sectors, can be written as

$$q_x(x;\varepsilon) = \frac{q^*\gamma\sigma^2(-c^*\theta - q^*)}{c^*(c^*\theta\psi + q^*)}\varepsilon,$$

which is negative in the reasonable calibration.

To complete the proof, the drift of x can be written as

$$\mu_x = x(1-x)\frac{q^*\gamma\sigma^2}{c^*\theta}\varepsilon - x^2\varepsilon\sigma^2 + x\varepsilon\sigma^2 + O(\varepsilon^2).$$

■

A.11 Lemma 3

This economy is a standard frictionless heterogeneous agent asset pricing model in a production economy.

Proof. We consider a first-order asymptotic expansion around the point $\epsilon = 0$, i.e., for each variable z, we want to find the functions $(z_0(w), z_1(w))$ such that $z(w;\epsilon) = z_0(w) + z_1(w)\epsilon + O(\epsilon^2)$. The zeroth-order coefficients are derived in the benchmark economy (see Lemma 1).

Let $\gamma_A = \gamma(1-\Delta\gamma)$, $\gamma_B = \gamma(1+\Delta\gamma)$, $\Delta\gamma > 0$, and

$$\Delta\gamma = \delta_\gamma\epsilon, \quad \kappa = \delta_\kappa\epsilon.$$

Aggregate risk aversion from (48) can be written as

$$\gamma(w;\epsilon) = \gamma\left(\frac{1-w}{1+\Delta\gamma} + \frac{w}{1-\Delta\gamma}\right)^{-1} = \gamma - (2w-1)\gamma\Delta\gamma + O(\epsilon^2).$$

In equilibrium, one of the FOCs gives $c_i = \xi_i$, thus the proof focuses on characterizing c_i.

Since the derivative of $q(w;0)$ (zeroth-order coefficient) with respect to w, is zero, the derivative of $q(w;\epsilon)$ with respect to w, namely $q_w(w;\epsilon)$, is first-order in ϵ. Similarly, the derivative of consumption-wealth ratio $c_i(w;\epsilon)$ with respect to w, namely $c_{i,w}(w;\epsilon)$ is also first-order in ϵ.

Meanwhile, the diffusion of w, σ_w is also first-order in ϵ because it is equal to zero at $\epsilon = 0$. Thus σ_q and σ_{ξ_i} are both of order $O(\epsilon^2)$ (see proof of Proposition 3 and 4 for expressions).

Then it follows that the market price of risk from (47) can be written as

$$\eta_t(w;\epsilon) = \gamma\sigma - \gamma\sigma(2w-1)\Delta\gamma + O(\epsilon^2).$$

Note that at $\epsilon = 0$, $\eta(w;0) = \gamma\sigma$, which is the market price of risk η^* from the benchmark economy in Lemma 1.

Then the risk exposure of investors from (37) can be written as

$$\begin{aligned}\sigma_A(w;\epsilon) &= \sigma + 2(1-w)\sigma\Delta\gamma + O(\epsilon^2),\\ \sigma_B(w;\epsilon) &= \sigma - 2w\sigma\Delta\gamma + O(\epsilon^2).\end{aligned}$$

Similarly, note that at $\epsilon = 0$, $\sigma_A(w;0) = \sigma_B(w;0) = \sigma$, which is the risk exposure from the benchmark economy in Lemma 1.

Using the above two expressions, the diffusion of w from (17) can be written as

$$\sigma_w(w;\epsilon) = w(1-w)2\sigma\Delta\gamma + O(\epsilon^2).$$

The interest rate, from firm's HJB, can be written as

$$r_t = \frac{A - \iota(g_t)}{q_t} + g_t + \mu_q + \sigma_q\sigma - \eta_t(\sigma_q + \sigma)$$

Use the market clearing condition for consumption goods, interest rate can be written as

$$\begin{aligned}r_t &= [wc_A + (1-w)c_B] + g_t + \mu_q + \sigma_q\sigma - \eta_t(\sigma_q + \sigma)\\ &= \psi\rho + \mu_t + (1-\psi)(r_t + \Phi_t) + g_t + \mu_q + \sigma_q\sigma - \eta_t(\sigma_q + \sigma)\end{aligned}$$

where the second line uses the optimal conditions for the consumption-wealth ratio (105) for investors A and B. Move the terms related to r_t to one side to solve for r_t, we have

$$r_t = \rho + \psi^{-1}[\mu_t + g_t + \mu_q + \sigma_q\sigma - \eta_t(\sigma_q + \sigma)] - (1 - \psi^{-1})\Phi_t.$$

where

$$\begin{aligned}
\mu_t &= w\mu_{\xi_A,t} + (1-w)\mu_{\xi_B,t} = O(\epsilon^2),\\
\Phi_{A,t} &= \frac{\gamma\sigma^2}{2} - \gamma\sigma^2(2w-1)\Delta\gamma + \frac{\gamma\sigma^2}{2}\Delta\gamma + O(\epsilon^2)\\
\Phi_{B,t} &= \frac{\gamma\sigma^2}{2} - \gamma\sigma^2(2w-1)\Delta\gamma - \frac{\gamma\sigma^2}{2}\Delta\gamma + O(\epsilon^2)\\
\Phi_t &= w\Phi_{A,t} + (1-w)\Phi_{B,t}
\end{aligned}$$

The first equation above is because μ_w is zero at $\epsilon = 0$, which means $\mu_w(w;\epsilon)$ is first-order in ϵ; and derivative $c_{i,w}(w;\epsilon)$ is first-order in ϵ. The drift terms of c and q are second-order in ϵ. Thus,

$$\begin{aligned}
r_t(w;\epsilon) = \rho &+ \psi^{-1}g^* - (1+\psi^{-1})\frac{\gamma\sigma^2}{2}\\
&+ \psi^{-1}(g_t - g^*) - \psi^{-1}\sigma(\eta_t - \eta^*)\\
&- (1-\psi^{-1})[w(\Phi_{A,t} - \Phi^*_{A,t}) + (1-w)(\Phi_{B,t} - \Phi^*_{B,t})] + O(\epsilon^2)\\
&= r^*\\
&+ \psi^{-1}(g_t - g^*) - \psi^{-1}\sigma(\eta_t - \eta^*)\\
&- (1-\psi^{-1})[w(\Phi_{A,t} - \Phi^*_{A,t}) + (1-w)(\Phi_{B,t} - \Phi^*_{B,t})] + O(\epsilon^2)
\end{aligned}$$

Following the first-order asymptotic expansion, write the price of capital as

$$q = q^* + q_1(w)\epsilon + O(\epsilon^2).$$

where q^* is the equilibrium price of capital from Lemma 1, and q_1 denotes the first-order coefficient of q with respect to ϵ.

From (25), $g = \frac{q-\phi}{\theta} - \delta$. Then growth rate can be written as

$$g = g^* + \frac{q_1}{\theta}\epsilon + O(\epsilon^2)$$

where $g^* = \frac{q^*-\phi}{\theta} - \delta$ is the equilibrium growth rate from Lemma 1.

Thus, the interest rate can be written as

$$r_t = r^* + \psi^{-1}\frac{q_1}{\theta}\epsilon + \frac{\gamma}{2}\sigma^2(1+\psi^{-1})(2w-1)\Delta\gamma + O(\epsilon^2)$$

After substituting the expression for r_t, the consumption-wealth ratios again from (105) can be written as

$$c_A = c^* + (1-\psi)\psi^{-1}\frac{q_1}{\theta}\epsilon + (1-\psi)[\frac{\gamma}{2}\sigma^2(\psi^{-1}-1)(2w-1) + \frac{\gamma\sigma^2}{2}]\Delta\gamma + O(\epsilon^2)$$
$$c_B = c^* + (1-\psi)\psi^{-1}\frac{q_1}{\theta}\epsilon + (1-\psi)[\frac{\gamma}{2}\sigma^2(\psi^{-1}-1)(2w-1) - \frac{\gamma\sigma^2}{2}]\Delta\gamma + O(\epsilon^2)$$

Plug the above expressions into the market clearing condition for consumption goods again, we can obtain an expression for $q_1(w)$ from

$$c^* + (\psi^{-1}-1)\frac{q_1}{\theta}\epsilon + (\psi^{-1}-1)\frac{\gamma}{2}\sigma^2(2w-1)\Delta\gamma + O(\epsilon^2) = \frac{A-\iota(g(q))}{q}$$

Using Lemma 1, $c^* = \frac{A-\iota(g(q^*))}{q^*}$, and $\iota = \frac{q^2-\phi^2}{2\theta}$, the expression for $q_1(w)$ can be simplified to

$$q_1(w) = \frac{(\psi-1)\theta q^*}{q^* + \psi\theta c^*}(2w-1)\frac{\gamma}{2}\sigma^2\delta_\gamma$$

Now all equilibrium objects can be expressed as functions of w and ϵ. For example, the growth rate of the economy can be written as

$$g(w;\epsilon) = g^* + \frac{(\psi-1)q^*}{q^* + \psi\theta c^*}(2w-1)\frac{\gamma}{2}\sigma^2\Delta\gamma + O(\epsilon^2)$$

Note when $\psi > 1$, $q_1(w)$ is increasing in w, i.e., the growth rate is increasing in w.

To complete the derivation, we have expressions for the drift term of w from (17). The second component can be written as

$$\begin{aligned}\mu_w(w;\epsilon) &= [(\sigma_A - \sigma_B)\eta + c_B - c_A] - (\sigma_A - \sigma_B)\sigma_R \\ &= (1 + \psi - \frac{2}{\gamma})\gamma\sigma^2\Delta\gamma + O(\epsilon^2),\end{aligned}$$

and the first component can be written as $\kappa(\bar{u} - w)$. ■

A.12 Lemma 4

This economy is a heterogeneous agent asset pricing model in a production economy with financial frictions. Lemma 3 shows that the frictionless optimal risk exposure σ_A, given by

$$\sigma_A^*(w;\epsilon) = \sigma + 2(1-w)\sigma\Delta\gamma + O(\epsilon^2),$$

is linearly decreasing in w. This implies that as the wealth share of type A investors decreases, their optimal risk exposure increases. Thus, financial constraints will bind for type A investors in equilibrium when their wealth share is sufficiently low.

Thus, in this lemma, on top of Lemma 3, we consider the case where the financial constraint is binding, i.e., the investor's risk exposure is constrained. We derive the equilibrium under this binding constraint. Thus the equilibrium in the presence of heterogeneous agents and financial constraint can be characterized as the following: Lemma 3 is the equilibrium when constraints do not bind, and this lemma is the equilibrium when the constraints bind for type A investors.

Proof. Consider $\sigma_A = \bar{\zeta}$. Following the same arguments that σ_q and σ_{ξ_i} are both of order $O(\epsilon^2)$, as in Lemma 3, we can solve for the risk exposure of type B investors under the binding constraint. Using the market clearing condition for risk exposure, the risk exposure of type B investors can be written as

$$\sigma_B(w;\epsilon) = \frac{\sigma - w\bar{\zeta}}{1-w} + O(\epsilon^2).$$

The market price of risk can be written as

$$\eta_t(w;\epsilon) = \gamma_B \frac{\sigma - w\bar{\zeta}}{1-w} + O(\epsilon^2).$$

We can write the following objects as

$$\begin{aligned}
\Phi_{A,t} &= \gamma_B \frac{(\sigma - w\bar{\zeta})\bar{\zeta}}{1-w} - \frac{\gamma_A \bar{\zeta}^2}{2} + O(\epsilon^2) \\
\Phi_{B,t} &= \frac{\gamma_B}{2}\left(\frac{(\sigma - w\bar{\zeta})}{1-w}\right)^2 + O(\epsilon^2) \\
\Phi_t &= w\Phi_{A,t} + (1-w)\Phi_{B,t}
\end{aligned}$$

Using the market clearing condition for goods, the interest rate can be expressed as

$$r_t = \rho + \psi^{-1}[\mu_t + g_t + \mu_q + \sigma_q\sigma - \eta_t(\sigma_q + \sigma)] - (1-\psi^{-1})\Phi_t.$$

Using same arguments from Lemma 3, we can write

$$\begin{aligned} r_t(w;\epsilon) &= \rho + \psi^{-1}(g_t - \eta_t\sigma) - (1-\psi^{-1})\Phi_t + O(\epsilon^2) \\ &= \rho + \psi^{-1}g^* - (1+\psi^{-1})\frac{\gamma\sigma^2}{2} \\ &+ \psi^{-1}\frac{q_1}{\theta}\epsilon - \psi^{-1}\eta_t\sigma - (1-\psi^{-1})\Phi_t + (1+\psi^{-1})\frac{\gamma\sigma^2}{2} + O(\epsilon^2) \\ &= r^* \\ &+ \psi^{-1}\frac{q_1}{\theta}\epsilon - \psi^{-1}\eta_t\sigma - (1-\psi^{-1})\Phi_t + (1+\psi^{-1})\frac{\gamma\sigma^2}{2} + O(\epsilon^2) \end{aligned}$$

Putting the above expression for the interest rate back in the firm's HJB, then we can solve for $q_1(w)$ as

$$q_1(w) = q^* \frac{2w^2\bar{\zeta}^2\psi - 2w^2\bar{\zeta}^2 - w\bar{\zeta}^2\psi + w\bar{\zeta}^2 - 2w\bar{\zeta}\sigma\psi + 2w\bar{\zeta}\sigma + \psi\sigma^2 - \sigma^2}{-g^*w\theta\psi + g^*\theta\psi + q^*w - q^* + r^*w\theta\psi - r^*\theta\psi + w\gamma\bar{\zeta}\theta\sigma\psi - \gamma\theta\psi\sigma^2}\theta\frac{\gamma}{2}\delta_\gamma.$$

Other equilibrium objects, including the drift term of w, can be expressed accordingly, as in Lemma 3. ■

B Numerical solution

The computation of equilibrium is reduced to the solution of a system of partial differential equations (PDEs) involving $(\xi_A(x,w), \xi_B(x,w), q_1(x,w), q_2(x,w))$.

The following procedure shows how to obtain a set of conditions involving (ξ_A, ξ_B, q_1, q_2) and its derivatives:

i. Using the optimal investment rate for firms in sector n from (25), I can express ι_n and g_n as functions of q_n.

ii. Using the expression for g_n from above, I can compute μ_x, drift of state variable x from (18). Note that the diffusion terms of x are exogenously given.

iii. Compute the variable z as a function of (q_1, q_2) from (27).

iv. Then I solve for the dynamics of state variable w, in particular the diffusion terms of w. Using the solutions for type A investors' risk exposures (43) from Proposition 6, and considering the margin constraints, I have

$$\sigma_A^{(1)} = \min(\sigma_A^{(1),*}, \bar{\zeta}^{(1)}), \quad \sigma_A^{(2)} = \min(\sigma_A^{(2),*}, \bar{\zeta}^{(2)}).$$

v. Using the market clearing conditions for the risk exposures again, I can compute $\sigma_B^{(1)}$ and $\sigma_B^{(2)}$ as follows:

$$\sigma_B^{(1)} = \frac{z\sigma_1 + Q_x x(1-x)\sigma_1 + [Q_w w(1-w) - w]\sigma_A^{(1)}}{Q_w w(1-w) + 1 - w},$$
$$\sigma_B^{(2)} = \frac{(1-z)\sigma_2 - Q_x x(1-x)\sigma_2 + [Q_w w(1-w) - w]\sigma_A^{(2)}}{Q_w w(1-w) + 1 - w}.$$

vi. Compute $\sigma_w^{(1)}$ and $\sigma_w^{(2)}$ using (17).

vii. Applying Itô's lemma, and using diffusion terms of x and w, compute the following:

$$\sigma_{q,n}^{(1)} = \frac{q_{n,x}}{q_n}\sigma_x^{(1)} + \frac{q_{n,w}}{q_n} w(1-w)\sigma_w^{(1)}; \quad \sigma_{q,n}^{(2)} = \frac{q_{n,x}}{q_n}\sigma_x^{(2)} + \frac{q_{n,w}}{q_n} w(1-w)\sigma_w^{(2)};$$
$$\sigma_{\xi,i}^{(1)} = \frac{\xi_{i,x}}{\xi_i}\sigma_x^{(1)} + \frac{\xi_{i,w}}{\xi_i} w(1-w)\sigma_w^{(1)}; \quad \sigma_{\xi,i}^{(2)} = \frac{\xi_{i,x}}{\xi_i}\sigma_x^{(2)} + \frac{\xi_{i,w}}{\xi_i} w(1-w)\sigma_w^{(2)}.$$

viii. Compute $(\eta^{(1)}, \eta^{(2)})$ using the expression for optimal risk exposures for type B investors from (37):

$$\eta^{(1)} = \gamma_B \sigma_B^{(1)} - \frac{1-\gamma_B}{1-\psi}\sigma_{\xi,B}^{(1)}, \quad \eta^{(2)} = \gamma_B \sigma_B^{(2)} - \frac{1-\gamma_B}{1-\psi}\sigma_{\xi,B}^{(2)}.$$

ix. Compute $(\boldsymbol{\sigma}_{R,1}, \boldsymbol{\sigma}_{R,2})$ by definition (31), which also defines $\boldsymbol{\Sigma}_R$.

x. Compute μ_w from (17), using optimal consumption-wealth ratios $c_i = \xi_i$ from (36) and above expressions.

xi. Applying Itô's lemma, compute $\mu_{q,n}$ for $n \in \{1,2\}$ and $\mu_{\xi,i}$ for $i \in \{A,B\}$ as follows:

$$\begin{aligned}
\mu_{q,n} &= \frac{q_{n,x}}{q_n}\mu_x + \frac{q_{n,w}}{q_n}\left[\kappa(\bar{\mu} - w) + w(1-w)\mu_w\right] \\
&\quad + \frac{1}{2}\frac{q_{n,xx}}{q_n}(\sigma_{x,1}^2 + \sigma_{x,2}^2) + \frac{1}{2}\frac{q_{n,ww}}{q_n}w^2(1-w)^2(\sigma_{w,1}^2 + \sigma_{w,2}^2) \\
&\quad + \frac{q_{n,xw}}{q_n}w(1-w)(\sigma_{x,1}\sigma_{w,1} + \sigma_{x,2}\sigma_{w,2}); \\
\mu_{\xi,i} &= \frac{\xi_{i,x}}{\xi_i}\mu_x + \frac{\xi_{i,w}}{\xi_i}\left[\kappa(\bar{\mu} - w) + w(1-w)\mu_w\right] \\
&\quad + \frac{1}{2}\frac{\xi_{i,xx}}{\xi_i}(\sigma_{x,1}^2 + \sigma_{x,2}^2) + \frac{1}{2}\frac{\xi_{i,ww}}{\xi_i}w^2(1-w)^2(\sigma_{w,1}^2 + \sigma_{w,2}^2) \\
&\quad + \frac{\xi_{i,xw}}{\xi_i}w(1-w)(\sigma_{x,1}\sigma_{w,1} + \sigma_{x,2}\sigma_{w,2}).
\end{aligned}$$

xii. Next I solve for interest rate r. Plugging the optimal consumption-wealth ratio (36) into the investors' HJB (35), I obtain

$$c_{i,t} - \psi\rho - \mu_{\xi_i,t} - (1-\psi)(r_t + \Phi_{i,t}) = 0,$$

where

$$\begin{aligned}
\Phi_{i,t} &= \boldsymbol{\sigma}_{i,t}'\boldsymbol{\eta}_t - \frac{\gamma_i}{2}\boldsymbol{\sigma}_{i,t}'\boldsymbol{\sigma}_{i,t} \\
&\quad + \frac{1}{2}\frac{1}{1-\psi}\left(\frac{1-\gamma_i}{1-\psi} - 1\right)\boldsymbol{\sigma}_{\xi_i,t}'\boldsymbol{\sigma}_{\xi_i,t} + \frac{1-\gamma_i}{1-\psi}\boldsymbol{\sigma}_{i,t}'\boldsymbol{\sigma}_{\xi_i,t}.
\end{aligned}$$

Plugging the optimal investment rate (25) into the firms' HJB (24), I obtain

$$\frac{A_n - \iota_{n,t}}{q_{n,t}} + \Omega_{n,t} - r_t = 0,$$

where

$$\Omega_{n,t} = g_{n,t} + \mu_{q_{n,t}} + \boldsymbol{\sigma}_{q_{n,t}}'\boldsymbol{\sigma}_{n,t} - \boldsymbol{\eta}_t'(\boldsymbol{\sigma}_{q_{n,t}} + \boldsymbol{\sigma}_{n,t}).$$

Using above equations in the goods market clearing condition,

$$wc_A + (1-w)c_B = z\frac{A_1 - \iota_1}{q_1} + (1-z)\frac{A_2 - \iota_2}{q_2}.$$

Rearranging the terms, I obtain the expression for the interest rate

$$r_t = \rho + \frac{1}{\psi}\mu_t + \left(\frac{1}{\psi} - 1\right)\Phi_t + \frac{1}{\psi}\Omega_t$$

where $\mu_t = w_t\mu_{\xi_A,t} + (1-w_t)\mu_{\xi_B,t}$, $\Phi_t = w_t\Phi_{A,t} + (1-w_t)\Phi_{B,t}$, and $\Omega_t = z\Omega_{1,t} + (1-z)\Omega_{2,t}$.

xiii. Plug all necessary equations into HJBs (24) and (35) to obtain the system of PDEs.

C Global solutions of alternative economies

C.1 Solutions of heterogeneous investors model with one sector

This section describes alternative economies that feature heterogeneous investors, and their solutions. They differ from the main model in that the investors have access to one risky asset: a claim to a Lucas 'tree' or a production firm. The demographics and preferences of investors are the same as in the main model.

Consumption-Based Model This section describes the consumption-based model with heterogeneous investors and one Lucas 'tree', and its solution.

Consumption is assumed to follow a standard geometric Brownian motion.

$$\frac{dC_t}{C_t} = \mu dt + \sigma dB_t, \tag{108}$$

where μ and σ are constants.

There is one state variable: the wealth share of intermediaries w. Denote the dividend yield of the claim to the Lucas 'tree' as F_t. The return of the risky asset can be written as

$$dR_t \equiv \frac{d(D/F)}{D/F} + F dt = \mu_R dt + \sigma_R dB_t, \tag{109}$$

where μ_R and σ_R denote the expected return and volatility of the risky asset, respectively. Applying Itô's lemma, we can express μ_R and σ_R as follows:

$$\mu_R = \mu + F - \mu_F - \sigma\sigma_F + \sigma_F^2, \tag{110}$$

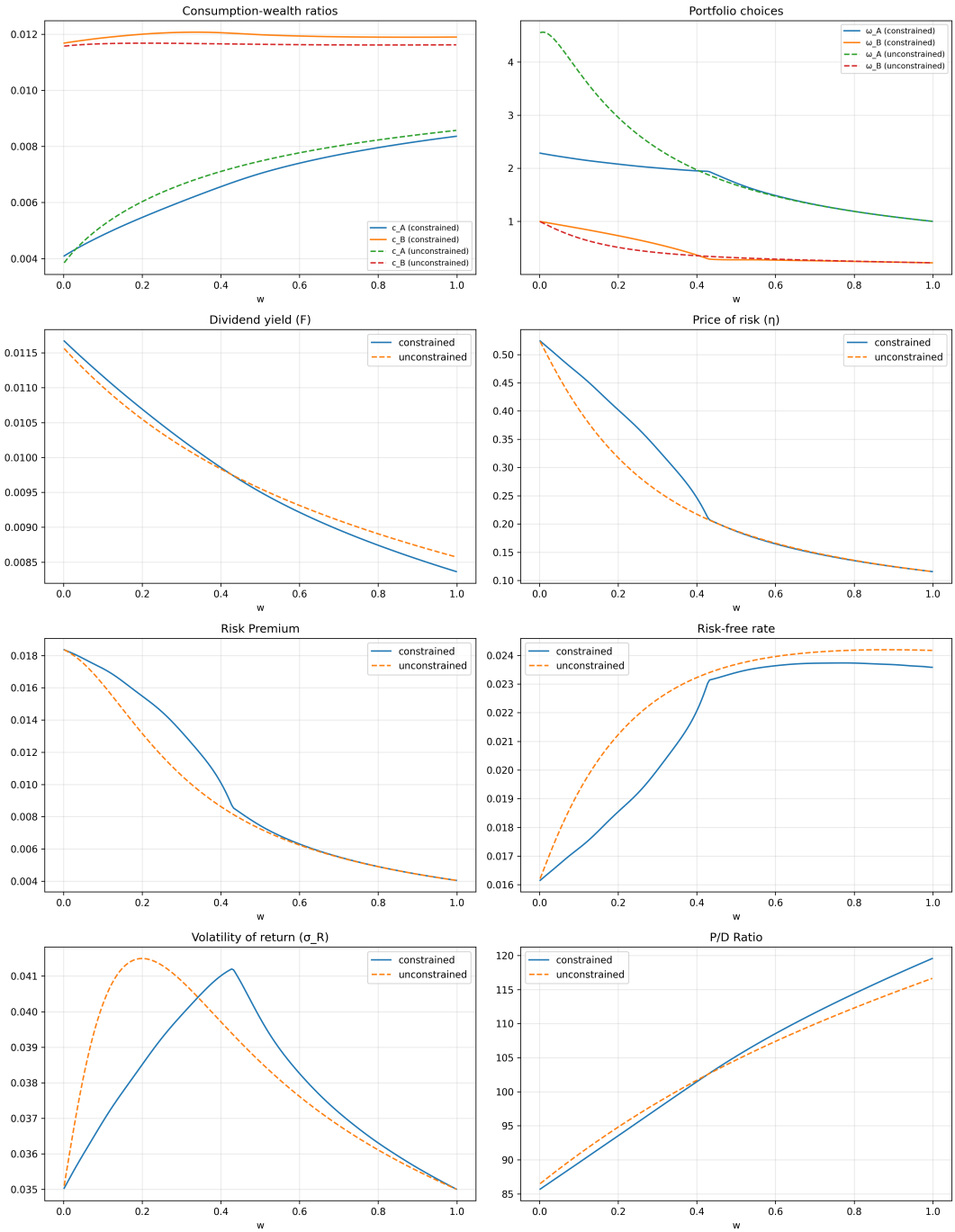
$$\sigma_R = \sigma - \sigma_F. \tag{111}$$

The equilibrium is characterized by two HJB equations for the two types of investors, the PDE that F_t satisfies, along with the market clearing conditions. The numerical solution is similar to that in the main model, except that there is only one state variable w. The calibration follows Kargar (2021) closely.

Figure C.1 shows the equilibrium results when the financial constraint is not binding for intermediaries, together with the results from the unconstrained economy for comparison. The solid lines represent the equilibrium with financial constraints, while the dashed lines represent the frictionless equilibrium. The panels on the first row display the consumption-wealth ratios $c_{i,t}$ and portfolio choices $\omega_{i,t}$ for investors of type i. The rest panels show the dividend yield F_t, price of risk η_t,risk premium $\lambda_{R,t}$, risk-free rate r_t, return volatility $\sigma_{R,t}$, and price-dividend ratio $1/F_t$. Each quantity is plotted against w_t, the wealth share of intermediaries,

which ranges from 0 to 1.

Figure C.1: Solutions of the consumption-based model



This figure shows the consumption-based model solution using the calibrated parameters from Kargar (2021). The solid lines represent the equilibrium with financial constraints, while the dashed lines represent the unconstrained equilibrium. The panels on the first row display the consumption-wealth ratios and portfolio choices for both types of investors. The rest panels show the dividend yield, price of risk,risk premium, risk-free rate, return volatility, and price-dividend ratio. Each quantity is plotted against w_t, the wealth share of intermediaries.

The frictionless economy is similar to Gârleanu and Panageas (2015).

The economy with financial constraints is similar to Kargar (2021), but with representative financial intermediaries.

Production-Based Model This section describes the production-based model with heterogeneous investors and one representative firm, and its solution.

The representative firm produces output using capital, and the same technology as the firms in the main model. The firm's capital stock evolves according to

$$\frac{dK_t}{K_t} = g_t\,dt + \sigma\,dB_t, \tag{112}$$

where σ is a constant, and g_t denotes the expected growth rate of capital at time t.

The equilibrium is characterized by three HJB equations for the firm and two types of investors, along with the market clearing conditions. The numerical solution is similar to that in the main model, except that there is only one state variable w. The calibration follows Silva (2020) closely.

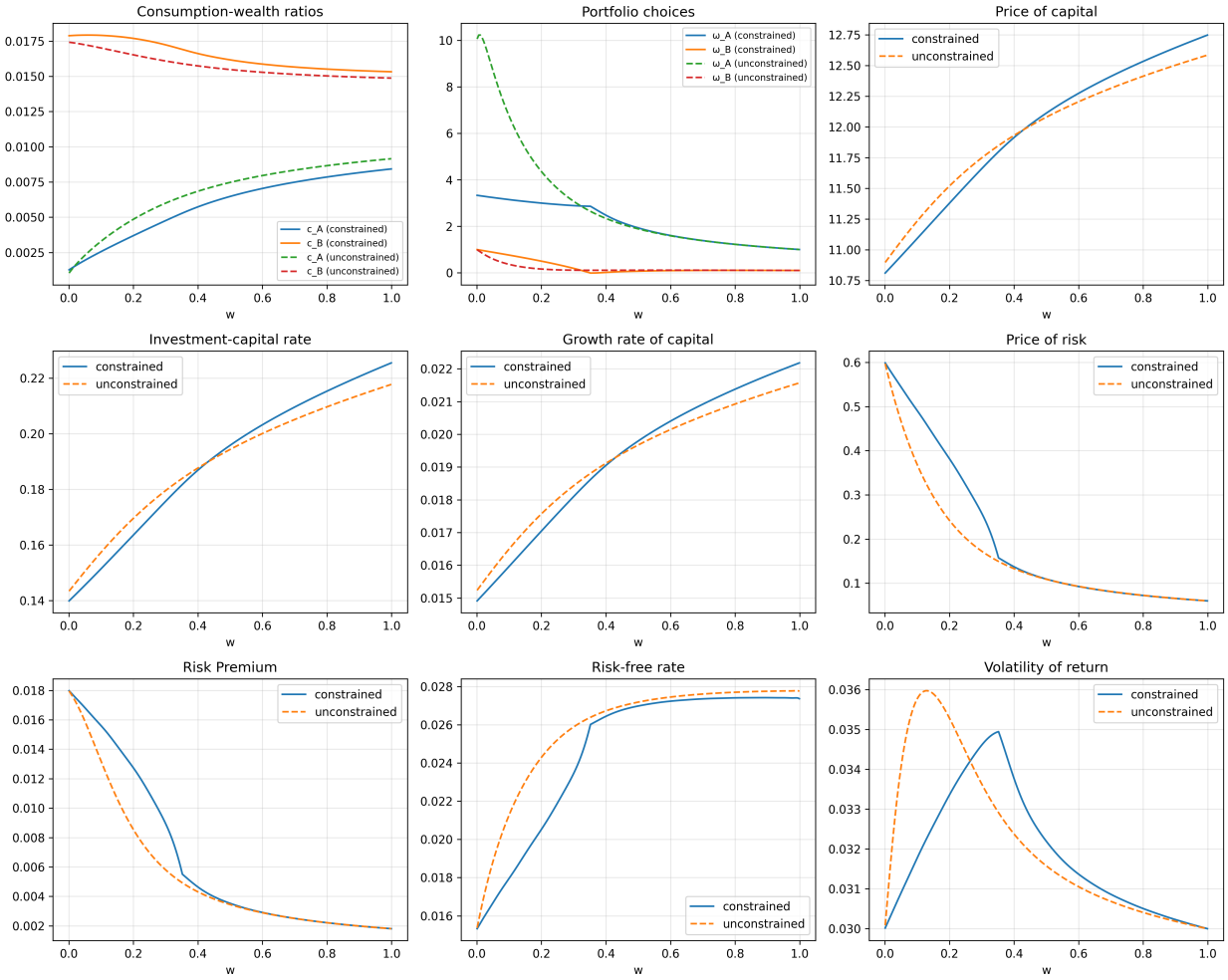
Figure C.2 shows the equilibrium results when the financial constraint is not binding for intermediaries, together with the results from the unconstrained economy for comparison.

C.2 Two-Sector Model

This section describes the economy from Eberly and Wang (2025) that features two sectors, and its solution. It differs from the main model in that there is a representative investor who has access to two risky assets: claims to dividends from firms in two sectors. The preference of investor is the same as that from Eberly and Wang (2025).

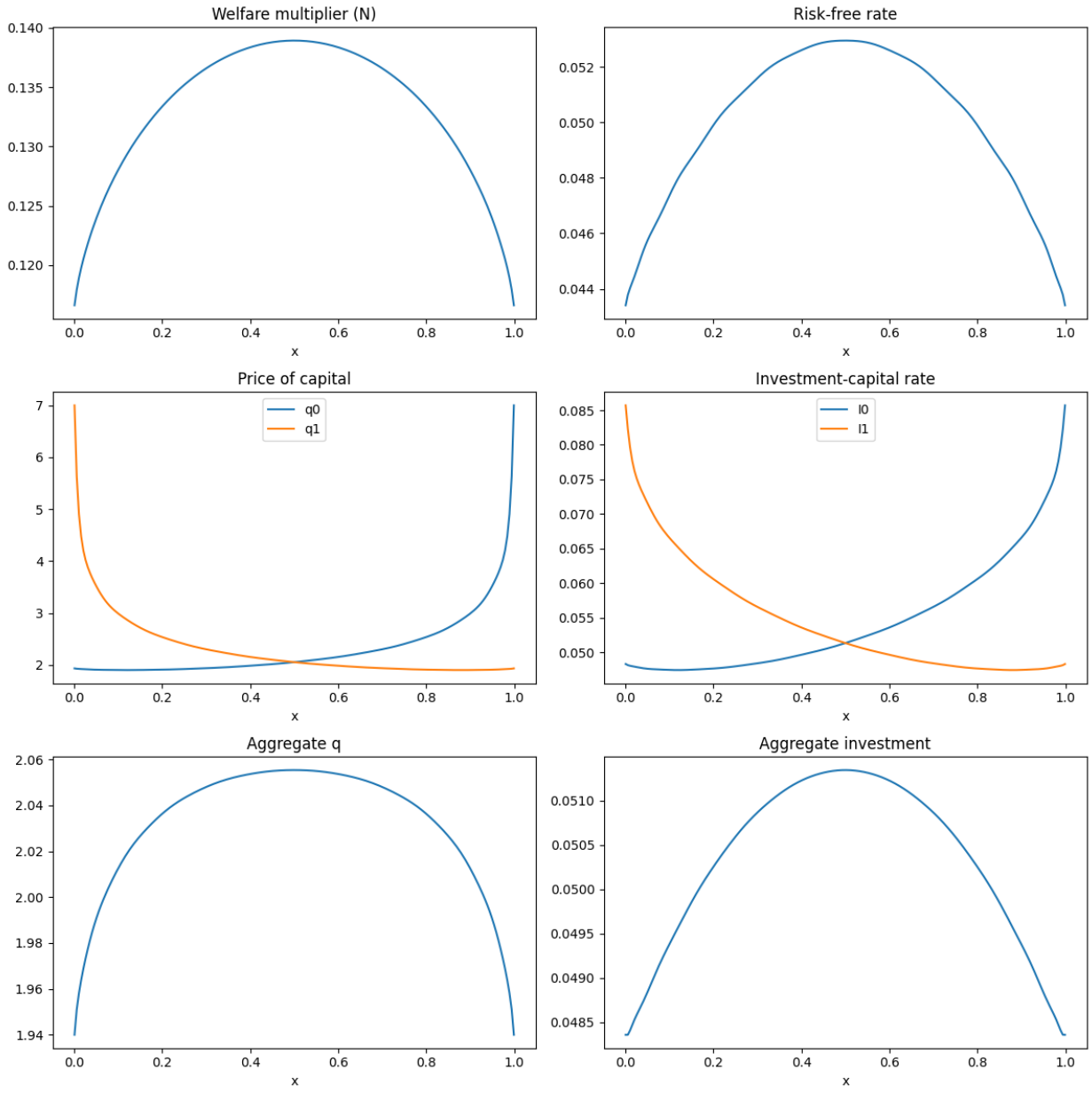
Due to the presence of two sectors and one representative investor, the model has one state variable: the capital share of sector 1, denoted as x. Here I follow Eberly and Wang (2025) and solve the model by invoking the welfare theorem. The social planner's problem can be characterized by a single HJB equation, and N denotes the welfare multiplier. Figure C.3 shows the equilibrium results when the two sectors are symmetric. The panels on the first row display the welfare multiplier of the representative investor and equilibrium risk-free rate r_t. The second row shows the prices of capital q and investment-capital rates for two sectors. The bottom row shows the aggregate price of capital and investment-capital rate. Each quantity is plotted against x_t, the capital share of sector 1, which ranges from 0 to 1.

Figure C.2: Solutions of the production-based model



This figure shows the production-based model solution using the calibrated parameters from Silva (2020). The solid lines represent the equilibrium with financial constraints, while the dashed lines represent the unconstrained equilibrium. The top and mid panels on the first row display the consumption-wealth ratios and portfolio choices for both types of investors. The rest panels show the price of capital, investment rate, growth rate of capital, price of risk,risk premium, risk-free rate, and return volatility. Each quantity is plotted against w_t, the wealth share of intermediaries.

Figure C.3: Solutions of the two-sector model



This figure shows the two-sector model solution using the calibrated parameters from Eberly and Wang (2025). The panels on the first row display the welfare multiplier of the representative investor and equilibrium risk-free rate r_t. The second row shows the prices of capital q and investment-capital rates for two sectors. The bottom row shows the aggregate price of capital and investment-capital rate. Each quantity is plotted against x_t, the capital share of sector 1.