Huggett Meets Epstein-Zin in Continuous Time

Hamilton Galindo Gil
Cleveland State University

Ji Huang
The Chinese University of Hong Kong

Yucheng Zhou
University of Illinois at Urbana-Champaign

September 6, 2025

Abstract

We extend the Huggett model to consider recursive preferences in continuous time. We then analyze the role of relative risk aversion (RRA) and elasticity of intertemporal substitution (EIS) in determining the equilibrium interest rate and the stationary wealth and consumption distributions. We show that EIS plays a crucial role in shaping wealth and consumption distribution at the aggregate and agent-type levels, while RRA plays a marginal role. Additionally, EIS has strong effects on interest rate and leverage compared to RRA. Our model is characterized by wealth and income heterogeneity among agents, incomplete markets, and a quantitative separation between RRA and EIS, providing a baseline framework for macro-finance models.

Keyword: heterogeneous agents; recursive preferences; incomplete markets; finite difference; labor income risk.

JEL: C63, E21, E43k

1 Introduction

Heterogeneity among agents and the quantitative separation between Relative Risk Aversion (RRA) and Elasticity of Intertemporal Substitution (EIS) are two fundamental components of modern macroeconomic and asset pricing models (e.g., Kaplan and Violante, 2018; Achdou et al., 2022; Pohl et al., 2021). However, incorporating both of these elements into a single framework complicates the search for model solutions. Consequently, researchers often simplify one of these components: either heterogeneity is reduced, for instance, to only two agents, or the quantitative separation of RRA and EIS is ignored by assuming Constant Relative Risk Aversion (CRRA) preferences (e.g, Wang, 1996; Pohl et al., 2021; Fernández-Villaverde et al., 2023).

In our study, we explicitly incorporate these two key modeling elements–heterogeneity and quantitative separation between RRA and EIS–into the continuous-time version of the Huggett (1993) model, which stands as a leading and tractable macroeconomic model. Specifically, we extend the continuous-time version developed by Achdou et al. (2022) by introducing recursive preferences from Epstein and Zin (1989). As a result, our model is characterized by incomplete markets, labor income represented by a two-state Poisson process, wealth and income heterogeneity among agents, and recursive preferences.

Equipped with this model, we proceed to analyze the roles of EIS and RRA in determining the existence of equilibrium, the optimal policy functions (consumption and saving), the shape of wealth and consumption distribution, and the equilibrium interest rate and leverage.

We first demonstrate that equilibrium is typically achieved when the RRA exceeds one and the EIS is less than one. This finding remains robust even when we modify our baseline calibration. It is important to note that this result is contingent upon our modeling assumptions, such as the dynamics of labor income and their calibration. This analysis leads us to explore the economic dynamics in a range of values for RRA and EIS.

Second, we show that the EIS plays a crucial role in shaping both aggregate and agent-level distributions. Specifically, variations in EIS lead to greater dispersion in wealth distribution at both aggregate and agent levels, with particularly pronounced effects among high-income agents. Similarly, changes in EIS lead to increased dispersion in consumption distribution at both levels, with particularly notable impacts among low-income agents. In contrast, we find that adjustments in RRA have marginal impacts on the wealth and consumption distribution among both low and high-income agents.

Third, we delve into the determination of interest rates and leverage in the economy. Our numerical simulations reveal that the EIS has a more significant impact on the equilibrium interest rate than RRA. This finding underscores the crucial role of EIS in shaping wealth and consumption distributions. Moreover, the leverage ratio, defined as total leverage over total income, shows a gradual decrease with RRA but a substantial increase with EIS. The meaningful impact of EIS on leverage is elucidated by its effects on interest rates.

These results underscore the distinct roles and size effects of EIS and RRA in determining the equilibrium interest rate and stationary distributions within heterogeneous-agent models, with a particular emphasis on the crucial role of EIS. Consequently, neglecting recursive preferences in such models would restrict our understanding of the effects of structural parameters on economic dynamics.

From a technical perspective, we solve the model by approximating derivatives using the finite difference method with an upwind scheme. Our approach is based on Achdou et al. (2022) but with some modifications. First, we multiply the Hamilton-Jacobi-Bellman (HJB) equation by $\theta = (1-1/EIS)/(1-RRA)$ to explicitly consider the effects of RRA and EIS greater or less than one on the finite difference technique. This subtle but important detail does not appear with CRRA preferences (i.e., $\theta = 1$). Neglecting this consideration could lead to a mistaken implementation of the upwind scheme. Second, the aggregator f in the recursive utility function has been evaluated such that the HJB equation becomes equivalent to one with CRRA preferences when $\theta = 1$. This assures us that our model has a solution for a CRRA case and provides us with a benchmark for comparison.

Our paper contributes to the heterogeneous-agent literature in macroeconomics and asset pricing (e.g., Huggett, 1993; Wang, 1996; Krusell et al., 2011; Longstaff and Wang, 2012; Gârleanu and Panageas, 2015; Schneider, 2022; Achdou et al., 2022; Fernández-Villaverde et al., 2023). We do so by embedding recursive preferences into a heterogeneous-agent model with incomplete markets. Specifically, we demonstrate that the Elasticity of Intertemporal Substitution (EIS) has stronger distributional effects than Relative Risk Aversion (RRA), and the quantitative distinction between EIS and RRA is relevant for understanding agents' optimal decisions, stationary distributions, interest rates, and leverage.

Our results align with findings from other studies, such as Schneider (2022), who show that EIS is crucial for explaining the term structure of interest rates. A closely related paper is Wang et al. (2016), which also explores recursive preferences in optimal decision-making. However, our study complements theirs in several key aspects. Firstly, while Wang et al. (2016) focuses on a representative agent, we consider heterogeneity in wealth and income among agents. Secondly, with a heterogeneous-agent model, we directly explore the effects of RRA and EIS on distributions from the model solution, distinguishing our approach from Wang et al. (2016). Thirdly, the endogeneity of the interest rate in our model allows us to study general equilibrium effects, whereas it is exogenous in Wang et al. (2016).

Overall, our findings underscore the importance of quantitatively distinguishing between RRA and EIS in heterogeneous-agent models.

The remainder of this paper is organized as follows. Section 2 describes the heterogeneous-agent model with Epstein and Zin (1989) preferences. Section 3 presents a quantitative analysis and Section 4 examines the distributional effects of RRA and EIS. Section 5 explores how leverage is determined in the economy and Section 6 concludes. Appendix A provides the proofs and derivations, Appendix B explains the implementation of finite difference and upwind scheme, and Appendix C shows convergency analysis using an alternative calibration.

2 The Model

We develop the model in two steps. We start by defining the set of assumptions in the economic setting section and, then, we define the equilibrium and the strategy to find it.

2.1 Economic Setting

2.1.1 Uncertainty, Information Structure, and Beliefs

Uncertainty. The uncertainty in the economy is represented by a filtered probability space $\{\Omega, \mathcal{F}, \mathbf{F}, \mathcal{P}\}$, in which a two-state Poisson process y is defined. Specifically, y represents the agent's income in every period changing randomly over time. This income can take two values $y \in \{y_1, y_2\}$, with $y_1 < y_2$. We also assume that the sample paths of y completely determine the true state of nature over time.

Information structure. The σ-field $\mathcal{F}_t$ is interpreted as representing *the information available at time* t. Furthermore, $\{\mathcal{F}_t^y\}$ is the augmented filtration generated by y.

Beliefs. The probability measure $\mathcal{P}$ is interpreted as representing the *agents' common beliefs*. Furthermore, all stochastic processes in this model are progressively measurable with respect to $\mathbf{F}$.

2.1.2 Goods Market

Supply side. As in Huggett (1993), We assume an endowment economy. Specifically, the agent receives an exogenous labor income y every period.
Demand side. The consumption space C_+ is defined as the set of positive, adapted consumption rate processes c that satisfy the integrability condition

$$\int_0^T c_{jt}^2 dt \leq \infty, \tag{1}$$

for every agent j in the economy.

2.1.3 Financial Markets

A. Financial Market Type. Financial markets are incomplete in the sense that there is only one riskless asset in the economy (a bond). Consequently, agents cannot fully hedge against the uncertainty in their income; that is, their income may be low (y_1) in some periods and high (y_2) in others

B. Financial Assets' Characteristics. The investment opportunity, represented by only one asset, is the riskless asset. This asset can equivalently be interpreted as a credit market where borrowers and lenders interact at the equilibrium interest rate r_t. Consequently, lenders represent agents demanding the riskless asset, while borrowers are agents supplying it. We assume that the net supply of this asset is zero.

2.1.4 Agents

Next, We model the agents' behavior that populated the economy. We split the agent modeling into five stages: the number of agents, the individual endowment, preferences, budget constraint (or wealth dynamic), and the agent's optimization problem.

A. The Number of Agents. The economy is populated by a continuous of agents who are heterogeneous in wealth a and labor income y.

B. Endowment. Every agent k is endowed with labor income y_t every period. Since y follows a continuous-time Poisson process, its conditional probabilities are defined as follows

$$P_{12} \triangleq \text{Prob}[y_{t+\Delta} = y_2 \mid y_t = y_1] \quad = \quad \lambda_1 \Delta + o(\Delta), \tag{2}$$

$$P_{11} \triangleq \text{Prob}[y_{t+\Delta} = y_1 \mid y_t = y_1] \quad = \quad 1 - \lambda_1 \Delta + o(\Delta), \tag{3}$$

where, P_{ij} is the (conditional) probability that the agent has a level of income y_j in the next period $t+\Delta$, given that he has y_i in period t. For instance, P_{11} represents the probability of an agent have the same income y_1 in period t and $t+\Delta$ (i.e., he stays in the state one). Furthermore, the probability of changing states is associated with an intensity parameter λ. Specifically, the income jumps from state 1 (y_1) to state 2 (y_2) with intensity λ_1 and vice versa with intensity λ_2.

It is worth noting that the intensity parameters, λ_1 and λ_2, and the income levels, y_1 and y_2, are exogenous. As Δ becomes infinitesimally small, conditional probabilities approach:

$$P_{11} \quad = \quad e^{-\lambda_1}, \quad P_{12} = 1 - P_{11} \tag{4}$$

$$P_{22} \quad = \quad e^{-\lambda_2}, \quad P_{21} = 1 - P_{22} \tag{5}$$

C. Preferences. The investor's utility function is given by the Stochastic Differential Utility (SDU), which is the continuous-time equivalent of *recursive utility* of Epstein-Zin. V_t is the SDU of agent j at time t expressed as[1]

$$V_{j,t} = E_t \left[\int_t^{\infty} f(c_{j,s}, V_{j,s}) ds \right] \tag{6}$$

where $f(\cdot)$ is a function called normalized aggregator defined as

$$f(c_j, V_j) = \frac{1}{1-\delta}(1-\gamma)V_j \times \left[c_j^{1-\delta} \left[(1-\gamma)V_j\right]^{-\frac{1-\delta}{1-\gamma}} - \rho \right], \quad \gamma \neq 1, \delta \neq 1, \tag{7}$$

[1] The literature places ρ (discount rate) differently into $f(\cdot)$. For instance, in Wang et al. (2016), ρ is a multiplicative term at the very beginning. In Schneider (2022), it is an additive term in the biggest parentheses as Eq. (7). Both modeling ways are similar since ρ consistently appears or not in the FOC. In the case of Wang et al. (2016) (multiplicative), ρ appears in the FOC, while in Schneider (2022)(additive), it does not. Both alternatives generate the HJB with CRRA preferences.

with $\theta = (1-\delta)/(1-\gamma)$. The relative risk aversion (RRA) and elasticity of intertemporal substitution (EIS) of agent j are γ and $1/\delta$, respectively. The subjective discount rate is represented by ρ. The well-known time-additive separable CRRA utility function is a special case of the SDU where RRA is the inverse of EIS; that is, $\gamma = \delta$, implying $\theta = 1$. In this case, $f(c_j, V_j) = c_j^{1-\gamma}/(1-\gamma) - \rho V_j$

D. Wealth Dynamic and Borrowing Constraint. Three forces drive the change in the agent's wealth a_{jt}: his income y_{jt}, his savings in the riskless asset a_{jt}, and his consumption c_{jt}. Therefore, the wealth dynamic of agent j is given by

$$da_{jt} = (y_{jt} + r_t a_{jt} - c_{jt})\, dt, \quad j = 1, 2, \tag{8}$$

E. Stochastic Optimal Control Problem. The stochastic optimal control problem of agent k, $\mathbf{P}$, is defined as

$$\max_{\{c_{jt}\}} E_t \left[\int_t^\infty f(c_{j,s}, V_{j,s}) ds \right], \tag{9}$$

subject to the wealth dynamic, borrowing constraint, and income process:

$$\begin{aligned} \text{Wealth dynamic of agent} &: \quad da_{jt} = (y_{jt} + r_t a_{jt} - c_{jt})\, dt & (10)\\ \text{Borrowing constraint} &: \quad a_{jt} \geq \underline{a} & (11)\\ \text{Income process} &: \quad y_{jt} \in \{y_{1t}, y_{2t}\} \text{ with } \lambda_1, \lambda_2 \text{ and } y_{1t} < y_{2t} & (12) \end{aligned}$$

Agents solve this problem by taking as given the interest rate $r_t, \forall t \geq 0$. We transform this problem into a stochastic dynamic programming one as Achdou et al. (2022).

2.2 The Equilibrium

We define equilibrium in the economy as follows.

Definition 1. *Equilibrium in this economy is defined as consumption processes $\{c_1, c_2\}$ and a price system $\{r\}$ such that at every period t: (i) agents maximize their expected discounted utility function taking as given the equilibrium interest rate; i.e., they solve the optimization problem* $\mathbf{P}$ *(Eq. 9 - 12), and (ii) all markets (bonds and goods market) clear. The bond market equilibrium condition is given by*

$$\underbrace{S(r)}_{\substack{\text{Total bonds}\\ \text{demand}}} \equiv \underbrace{\int_{\underline{a}}^\infty a dG_1(a,t)}_{\substack{\text{bonds demand from agents}\\ \text{who have income } y_1}} + \underbrace{\int_{\underline{a}}^\infty a dG_2(a,t)}_{\substack{\text{bonds demand from agents}\\ \text{who have income } y_2}} = \underbrace{B}_{\substack{\text{fixed}\\ \text{bonds supply}}}, \tag{13}$$

where the aggregate bond demand is represented by $S(r)$ and the aggregate supply is fixed and equals B. We assume the bond is in zero net supply and then $B = 0$. Furthermore, $G_j(a,t)$ is the cumulative join distribution function (CDF) for agent type j at period t (income-wealth distribution), and $d\tilde{G}_j(a) = g_j(a)da$. Furthermore, $g_j(a,t)$ represents the *density of the* joint distribution *of* y_j *and* a *in period* t*. The equilibrium condition for the goods market is given by*

$$c_{jt} + s_{jt} = y_{jt} + r_t a_{jt}, \quad j \in \{1, 2\}, \tag{14}$$

where s_{jt} *is the saving of agent* j *and it is equals to the change of his wealth* da/dt*.*

2.2.1 Finding the Equilibrium

We are interested in the *stationary* equilibrium; that is, an equilibrium in which the joint income-wealth distribution is invariant over time and the corresponding optimal consumption, saving, and the equilibrium interest rate.

The Strategy. Finding the *stationary* equilibrium can be split into three steps. The first one is the stochastic dynamic programming problem. This step transforms the stochastic optimal control problem **P** (Eq. 9 - 12) into the HJB equation for every agent j resulting in two PDEs of the agent's value function: one for each income type. Then, We calculate the first-order conditions (FOCs) with respect to consumption. The second step is to obtain the PDE of the joint income-wealth density $g_j(a)$, called the Kolmogorov Forward (KF) (or Fokker-Planck) equation. The solution of this PDE gives us the stationary distribution. The third step is to represent the model as a *system of partial differential equations (PDEs)* (HJB and KF equations) with equilibrium conditions, first-order conditions, and constraints.

We solve these PDEs in Section 3 using the finite difference method with the upwind scheme. Heuristically, the algorithm is as follows: given an initial value function and interest rate, the HJB equation is solved. Then, using the policy functions, we solve the Kolmogorov Forward equation, providing a density function. Next, the interest rate is calculated using the equilibrium in the bond market and the previous density function. Finally, if this interest rate is close enough to the *initial* value, we find the *stationary* equilibrium. We provide the proof of the following lemmas in Appendix A.

Step 1. *The Dynamic Programming Problem.* In this step, We obtain the HJB equation for agent j and the first-order conditions. Since we have one state variables in the economy, the agent's wealth a, the HJB equation is a PDE of the agent-j's value function related to only one state variable.

(a) The HJB Equation. Let $V_j(t, a)$ be the value function at date t of agent type $j \in \{1, 2\}$, which depends on his individual wealth a, defined as

$$V_j(t, a) = \sup_{\{c_{jt}\}} E_t \left[\int_t^{\infty} f(c_{j,s}, V_{j,s}) ds \right], \tag{15}$$

The HJB equation for agent j (with constraints) is defined as follows.

Lemma 2.1. *The HJB equation of $V_j(t,a)$ for $j = \{1,2\}$ with an initial condition is given by*

$$0 = \max_{\{c\}} \left\{ f(c_j, V_j) + V_j'(a)(y_j + ra - c) + \lambda_j(V_{-j}(a) - V_j(a)) \right\}, \tag{16}$$

where $-j$ represents the other agent type (with the other labor income level). For instance, if $j = 1$ (i.e, $y = y_1$), then $-j = 2$ (i.e., $y = y_2$).

(b) First Order Conditions. The static optimization problem from the HJB equation (16) is given by

$$\max_{\{c_{jt}\}} \{\Psi\}, \tag{17}$$

where Ψ represents the elements inside braces of the HJB equation (16). Then, the first-order condition is given by

$$c_j \quad : \quad \frac{\partial f(c_j, V_j)}{\partial c_{kt}} - V_j'(a) = 0 \longrightarrow ((1-\gamma)V_j)^{1-\theta} \, c_j^{-\delta} = V_j'(a), \tag{18}$$

where $\theta = (1-\delta)/(1-\gamma)$.

(c) The Optimal HJB Equation. Considering the optimal control variable (18) in the HJB equation (16), the operation *max* is not longer necessary. Then, the optimal HJB equation for agent type $j \in \{1,2\}$ is given by

$$0 \quad = \quad f(c_j, V_j) + V_j'(a)\,(y_j + ra - c) + \lambda_j\,(V_{-j}(a) - V_j(a)) \tag{19}$$

with

$$\text{First Order Condition (FOC)} \quad : \quad ((1-\gamma)V_j)^{1-\theta} \, c_j^{-\delta} = V_j'(a) \tag{20}$$

$$\text{Wealth dynamic of agent} \quad : \quad da_{jt} = (y_{jt} + r_t a_{jt} - c_{jt})\, dt \tag{21}$$

$$\text{Borrowing constraint} \quad : \quad a_{jt} \geq \underline{a} \tag{22}$$

$$\text{Income process} \quad : \quad y_{jt} \in \{y_{1t}, y_{2t}\} \text{ with } \lambda_1, \lambda_2 \text{ and } y_{1t} < y_{2t} \tag{23}$$

Step 2. *The Kolmogorov Forward Equation.* In this step, we find the PDE of the joint income-wealth density $g_j(a)$ for agent type $j \in \{1,2\}$. To do so, we start find the change of the cummulated wealth-income joint distribution function $G_j(a,t)$ over a small period Δ in a discrete-time framework; that is, $(G_j(a, t+\Delta) - G_j(a,t))/\Delta$. Then, we take limit when Δ tends to zero to find the dynamics of $G_j(a,t)$. Finally, we use the fact that $\partial_a G_j(a,t) = g_j(a,t)$ and then set up $\partial g_j(a,t)/\partial t = 0$ since we are interested in the *stationary density*. This last step allows us to find the Kolmogorov Forward Equation, stated in the following Lemma.

Lemma 2.2. *Given the optimal value function from the HJB equation and the policy functions $\{c_j(a), s_j(a)\}$ for $j \in \{1,2\}$, the the stationary wealth-income joint density $g_j(a)$ is the solution of the following PDE, called the Kolmogorov Forward Equation.*

$$0 = -\partial_a[s_j(a) g_j(a)] - \lambda_j g_j(a) + \lambda_{-j} g_{-j}(a), \tag{24}$$

where $\partial_a X \equiv \partial X/\partial a$ and optimal saving $s_j = y_j + ra - c_j$.

Step 3. *The System of PDEs.* The model is represented by two PDEs (the HJB and FK equations) with equilibrium and optimality conditions along with constraints. This system is given by

$$\begin{aligned} \textbf{HJB Eq.} &: \quad 0 = f(c_j, V_j) + V_j'(a)\left(y_j + ra - c\right) + \lambda_j\left(V_{-j}(a) - V_j(a)\right) &(25)\\ \textbf{KF Eq.} &: \quad 0 = \partial_a[s_j(a)g_j(a)] - \lambda_j g(a) + \lambda_{-j} g_{-j}(a) &(26)\end{aligned}$$

with

$$\begin{aligned} \text{FOC} &: \quad \left((1-\gamma)V_j\right)^{1-\theta} c_j^{-\delta} = V_j'(a) &(27)\\ \text{Wealth dynamic of agent} &: \quad da_{jt} = \left(y_{jt} + r_t a_{jt} - c_{jt}\right) dt &(28)\\ \text{Borrowing constraint} &: \quad a_{jt} \geq \underline{a} &(29)\\ \text{Income process} &: \quad y_{jt} \in \{y_{1t}, y_{2t}\} \text{ with } \lambda_1, \lambda_2 \text{ and } y_{1t} < y_{2t} &(30)\\ \text{Marker clear condition} &: \quad S(r) \equiv \int_{\underline{a}}^{\infty} a dG_1(a) + \int_{\underline{a}}^{\infty} a dG_2(a) = B &(31)\\ \text{Aggregation of densities} &: \quad \int_{\underline{a}}^{\infty} g_1(a)da + \int_{\underline{a}}^{\infty} g_2(a)da = 1 &(32)\\ \text{Normalized aggregator} &: \quad f(c_j, V_j) = \frac{1}{1-\delta}(1-\gamma)V_j\left[c_j^{1-\delta}\left[(1-\gamma)V_j\right]^{-\frac{1-\delta}{1-\gamma}} - \rho\right] &(33)\end{aligned}$$

where $B = 0$ and the distribution function $G_j(a,t)$ is related to the density function $g_j(a,t)$ by $G_j(a,t) = g_j(a,t)da$ for $j \in \{1, 2\}$.

3 Quantitative Analysis

3.1 The Numerical Solution Method

We follow the same numerical procedure to solve a heterogeneous agents model with only idiosyncratic shock proposed by Achdou et al. (2022). Specifically, we use a bisection procedure to find the interest rate at the stationary equilibrium. The steps are as follows: First, We define an initial guess $r^{(0)} = 0.03$ and iterate to obtain $r^{(n)}$ for each iteration $n = 0, 1, 2, 3, \ldots, n_{\max}$. Second, given $r^{(n)}$, we solve the HJB equation (25) using the finite difference method with the upwind scheme. The result is the optimal consumption $c_j^n(a)$ and savings $s_j^n(a)$ for $j \in 1, 2$. Third, given $s_j^n(a)$, we solve the KF equation (26) for $g_j^n(a)$ using the finite difference method. Fourth, given $g_j^n(a)$, we compute the net supply of bonds $S(r^{(n)})$, which must be equal to zero (Eq. 31). We update the interest rate based on the following rule: if $S(r^{(n)}) > 0$, we decrease the interest rate in the next iteration ($r^{(n+1)} < r^{(n)}$), while if $S(r^{(n)}) < 0$, we increase the interest rate in the next iteration ($r^{(n+1)} > r^{(n)}$). The algorithm terminates when $r^{(n+1)}$ and $r^{(n)}$ are almost the same. In this case, the stationary equilibrium is represented by $\left(r^{(n)}, V_1^{(n)}, V_2^{(n)}, g_1^n, g_2^n\right)$. Appendix B describes the details of the numerical procedure.

3.2 Parameter Values

We then calibrate the model using the parameter values from heterogeneous-agent literature (see Table 1). We set up the subjective discount rate ρ equals to 0.05 to obtain a discount factor equals to 0.9512, standard in this literature (e.g., Chan and Kogan, 2002; Fernández-Villaverde et al., 2023). We choose the RRA (γ) and EIS ($\psi = 1/\delta$) from the space of values that allows convergency of the HJB equation and equilibrium. Particularly, we show in the next section that the model has a solution and hence the equilibrium exists for combinations of the values of $\gamma \in [1.5, 8.4]$ and $\psi \in [0.2, 0.6]$. For the baseline calibration, We choose $\gamma = 3$ and $\psi = 0.2$.

We closely follow Fernández-Villaverde et al. (2023) in calibrating the intensity of jumps between states, denoted as λ_j, and the income levels, denoted as y_j. First, states one and two can be interpreted as representing the unemployment and employment situations, respectively. Then, the pair (λ_1, λ_2) represents the transition rates between these two states. We calculate these parameters considering that the unemployment rate $\lambda_2/(\lambda_1+\lambda_2)$ is 0.05 and the job finding rate λ_1 is 0.986 annually. Moving to the income levels, we calculate y_1 and y_2 using two sources. First, we use a normalization condition: $1 = (\lambda_2/(\lambda_1 + \lambda_2))\, y_1 + (\lambda_1/(\lambda_1 + \lambda_2))\, y_2$. Second, we assume that $y_1 = 71\% y_2$.

3.3 Convergence and Equilibrium

Based on our baseline calibration, shown in Table 1, we proceed to evaluate the cases in which the model has an equilibrium. Specifically, we inspect the values of γ (RRA) and ψ (EIS) for which the HJB equation converges and the model has an equilibrium interest rate. To do so, we create a grid of values of $\gamma \neq 1$ and $\psi \neq 1$ between 0.1 and 10, increasing

Table 1: Baseline Parameter Values

Parameters	Symbol	Value
Subjective discount rate	ρ	0.05
Relative risk aversion (RRA)	γ	3
Elasticity of intertemporal substitution (EIS)	$1/\delta$	0.2
Intensity to jump between states	$\{\lambda_1, \lambda_2\}$	$\{0.986, 0.052\}$
Borrowing limit	$\underline{a}$	-0.15
Income level	$\{y_1, y_2\}$	$\{0.71, 1.015\}$

by 0.1. For instance, γ takes values from $\{0.1, 0.2, 0.3, \cdots, 0.9, 1.1, \cdots 9.9, 10\}$. We then solve the model for each pair combination (γ, ψ), which are 9801 in total. We restrict our search for an equilibrium interest rate between one and four percent, since a stability condition suggests that $r < \rho = 5\%$.

We first evaluate in what space of pairs (γ, ψ) the HJB converges. Our simulation shows that the model has a HJB equation convergence for 12.5% of the total pairs (γ, ψ), as shown on the left graph of Figure 2. Most of these values can be split into two subsets: $(\gamma > 1, \psi < 1)$ and $(\gamma < 1, \psi > 1)$. Few pairs of $(\gamma > 1, \psi > 1)$ generate convergence in the HJB equation.

Second, our simulations also show that for 203 pairs out of a total of 9801, the model has a solution, that is, convergence of the HJB equation and equilibrium interest rate (see the right graph of Figure 2). This result suggests that 2.1% of the total pairs (γ, ψ) evaluated generate equilibrium. As we can observe, the set of values of (γ, ψ) ensuring equilibrium constitutes a subset of the space where (γ, ψ) guarantees convergence of the HJB equation. Specifically, the model equilibrium is assured by some combinations of the values of $\gamma \in [1.5, 8.4]$ and $\psi \in [0.2, 0.6]$.

This analysis has an important implication for heterogeneous-agent models: the equilibrium requires that $\gamma > 1$ and $\psi < 1$. This requirement is robust even when we change the calibration, as shown in Appendix C. In our robustness exercise, we set $\lambda_1 = \lambda_2 = 0.6931$ to get a probability of 30.12% to keep the agent in the same state and, based on Achdou et al. (2022), we consider labor income as $y_2 = 2y_1 = 0.2$. This exercise confirms that the equilibrium typically requires $\gamma > 1$ and $\psi < 1$, as previously observed in our initial analysis.

We then explore the behavior of the interest rate in the space of (γ, ψ), where the equilibrium exists. The left graph of Figure 2 plots the equilibrium interest rate for each pair of (γ, ψ), while the right graph of Figure 2 shows the equilibrium interest rate versus γ (RRA) for each value of ψ (EIS). These graphs provide insights into the relationship between the interest rate, RRA, and EIS. First, the interest rate decreases monotonically with RRA, regardless of the level of EIS. Second, the interest rate increases with EIS. Third, the marginal effect of EIS on interest rates decreases across different values of EIS. For instance, fixing $\gamma = 3$, increasing ψ from 0.2 to 0.3 has a larger effect on the interest rate than increasing it from 0.4 to 0.5.

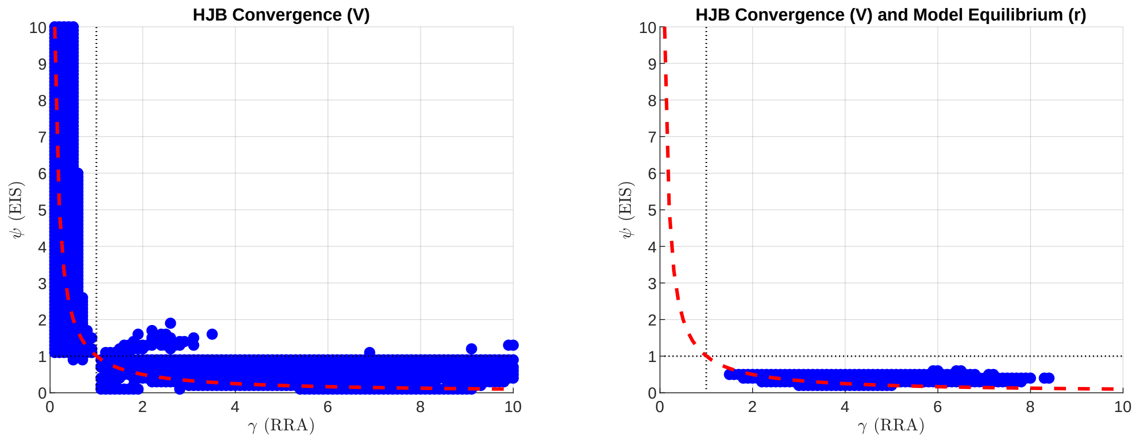

Figure 1: **Model Convergence and Equilibrium.**

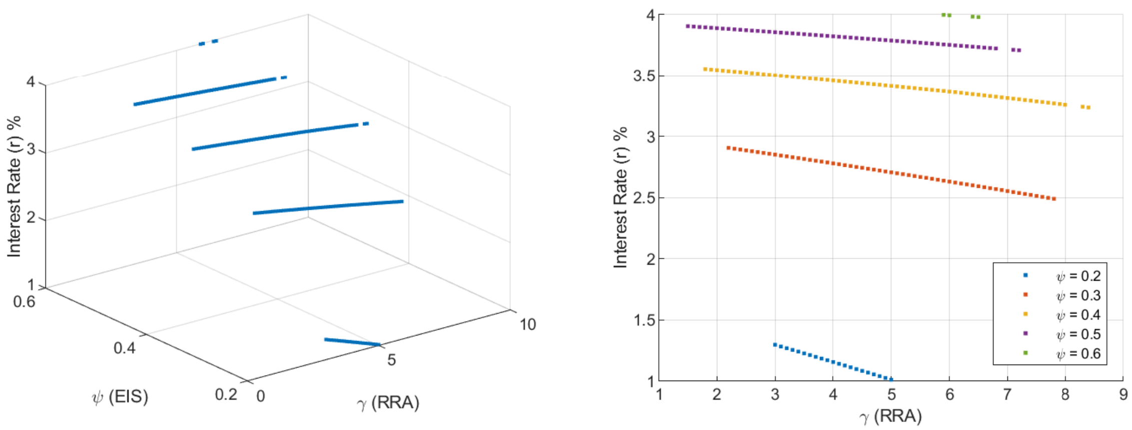

Figure 2: **Interest Rate and its Relationship with RRA and EIS.**

Discussion about $\psi < 1$. It might seem surprising that Huggett's model with recursive preferences does not yield equilibria for values of the Elasticity of Intertemporal Substitution (EIS) greater than one (based on our calibration and model assumptions), particularly considering the success of long-run risk models in explaining asset pricing puzzles, often assuming an EIS greater than one (Bansal and Yaron, 2004; Pohl et al., 2021). However, the literature lacks consensus on the precise value of ψ (see Thimme, 2017, and references therein), with estimates of $\psi < 1$ emerging across various contexts such as multiple consumption goods or limited participation in stock markets. Since our study does not aim to elucidate financial market dynamics, our findings align with this latter strand of literature.

Another crucial consideration is that recursive preferences are also intertwined with attitudes towards uncertainty resolution. Our convergence results suggest the existence of equilibrium for both types of preferences: early resolution ($\gamma > 1/\psi$) and late resolution

($\gamma < 1/\psi$) of uncertainty, stemming from labor income risk. This flexibility enables the model to remain consistent with evidence favoring a preference for early resolution of uncertainty (Epstein et al., 2014; Ai et al., 2023).

3.4 Optimal Consumption and Savings

Having established the parameter space (γ, ψ) wherein the model yields a solution, we now delve into examining the policy functions. Figure 3 depicts the optimal consumption and saving behaviors for each agent type $j \in \{1, 2\}$ under both the baseline calibration and a variation in γ (or ψ), while holding the other parameter constant.

The upper panel of Figure 3 shows the consumption and saving behaviors for the baseline calibration ($\gamma = 3$ and $\psi = 0.2$), contrasting them with the outcomes of adjusting γ from 3 to 5 while keeping ψ constant at 0.2. Interestingly, movement in the Relative Risk Aversion (RRA) parameter (γ) has marginal effects on both consumption and saving rules. This contrasts strongly with the effect of the Elasticity of Intertemporal Substitution (EIS) parameter (ψ), depicted in the bottom panel of Figure 3. Specifically, this panel reveals that when EIS (ψ) increases from 0.2 to 0.5, consumption and saving respond more strongly than to changes in RRA (γ). Moreover, the low-income level agent experiences a greater increase in his consumption rule (c_1) compared to the high-income level agent (c_2).

Why does it explain these results? The equilibrium interest rate plays a crucial role. Initially set at 1.3% under the baseline calibration, it experiences a minor decrease to 1.01% when γ rises from 3 to 5. However, when ψ escalates from 0.2 to 0.5, the interest rate sees a significant surge, soaring to 3.9%. Consequently, changes in the EIS have a more pronounced impact on the interest rate than alterations in RRA, and hence on optimal consumption and saving.

3.5 Wealth and Consumption Distributions

We then inspect the stationary wealth and consumption distributions generated by the model and the distributional effects of changes of γ and ψ. Figure 4 shows the effects of changing γ on wealth and consumption distribution for both agent types (low and high income), while Figure 5 shows the effects of changing ψ on the same variables for both agents. We explore first the stationary distributions.

These figures reveal the following regarding wealth distribution. First, there is a concentration of low-income agents at the borrowing limit ($\underline{a} = -0.15$) in their wealth distribution. Second, it seems that most low-income agents are borrowers ($a < 0$). Third, upon inspecting high-income agents, their wealth distribution is negatively skewed, suggesting that on average these agents are net lenders ($a > 0$).

Exploring the consumption distribution, these figures suggest the following. First, there is also a concentration of low-income agents at the minimum level of consumption, which is related to the concentration at the borrowing limit in wealth distribution. Second, the consumption of high-income agents is higher than that of low-income agents for all values of wealth.

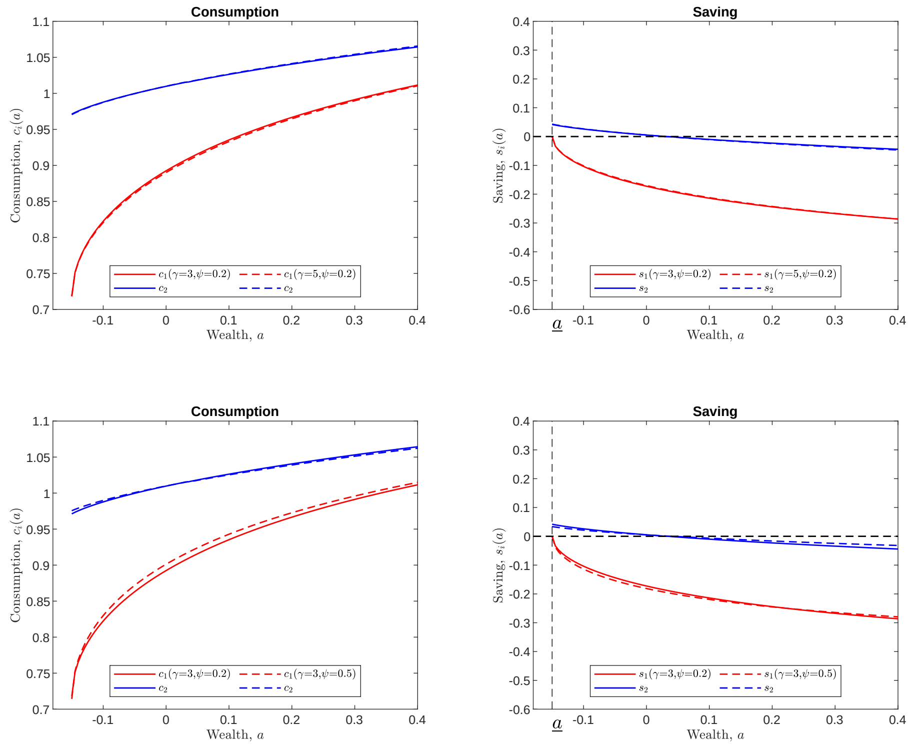

Figure 3: **Optimal Consumption and Saving.** This figure illustrates the optimal policy functions for consumption and saving for both low-income (y_1) and high-income (y_2) type agents. The baseline calibration assumes $\gamma = 3$ and $\psi = 0.2$. The graphs in the upper panel depict the policy functions for consumption and saving when γ increases from 3 to 5 while holding ψ constant at 0.2. Conversely, the graphs in the lower panel illustrate the policy functions for consumption and saving when ψ increases from 0.2 to 0.5 while maintaining γ at 3.

Having explored the stationary distributions generated by the model, we move on to analyze the effects of γ and ψ. Surprisingly, changes in γ (Figure 4) have marginal affects on the wealth and consumption distribution of both agent types. In contrast, variations in ψ have non-trivial distributional effects. Specifically, an increase in ψ from 0.2 to 0.5 shifts the wealth distribution of the high-income agent to the right, suggesting a higher median but higher dispersion. A similar effect is observed in their consumption distribution, although it is less pronounced.

These results highlight the *different* effects of RRA and EIS on the optimal decisions of agents and on their stationary distributions. Therefore, it is of first-order importance to consider them independently in a heterogeneous-agent model. This can be achieved when preferences are modeled recursively, as in Epstein-Zin preferences. However, using constant relative risk aversion (CRRA) preferences does not seem to accurately capture the distributional effects of RRA and EIS.

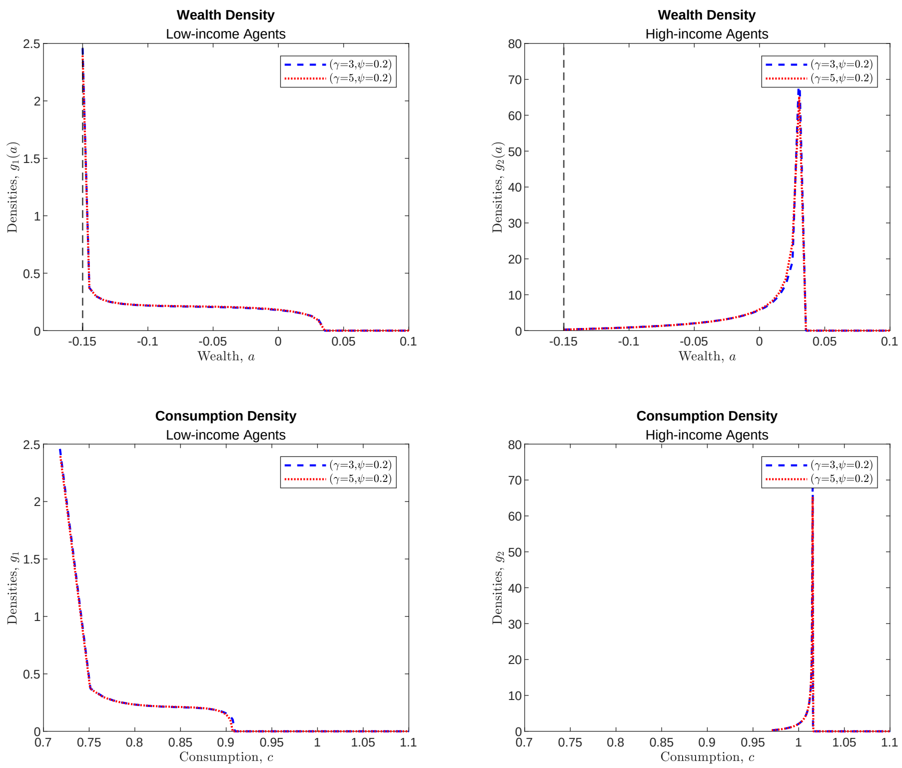

Figure 4: **Wealth and Consumption Distribution ($\Delta\gamma$).** This figure shows the distributional effects of an increase of RRA (γ) from 3 to 5, for both agent types.

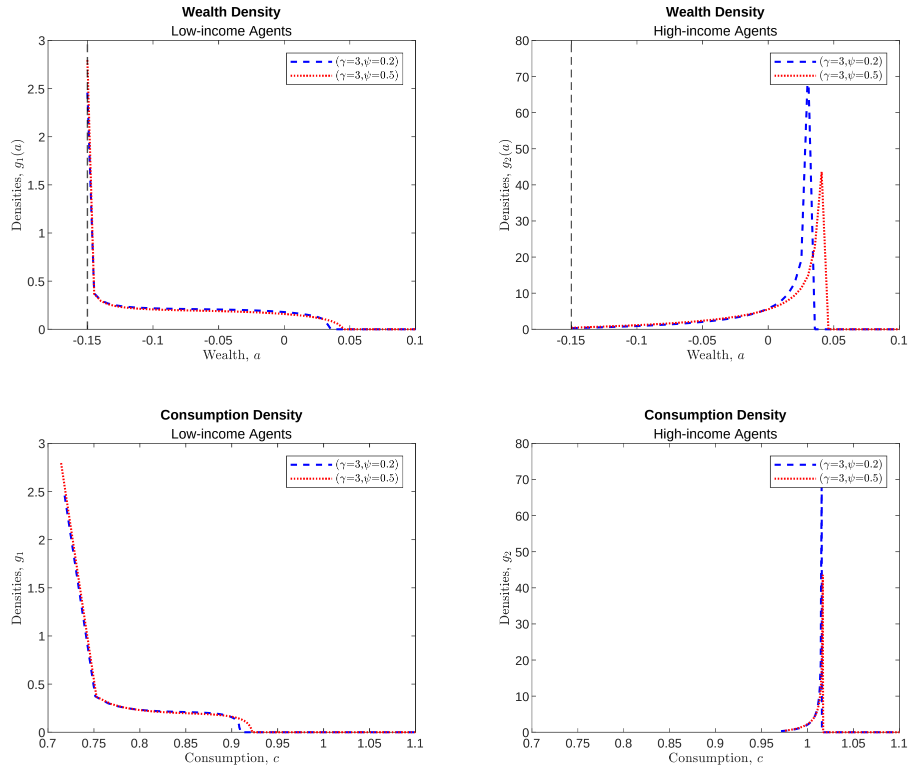

Figure 5: **Wealth and Consumption Distribution ($\Delta\psi$).** This figure shows the distributional effects of an increase of EIS (ψ) from 0.2 to 0.5, for both agent types.

4 Distributional Effects of RRA and EIS

In this section, we delve deeper into the distributional effects of RRA and EIS, building upon the graphical analysis presented in the previous section. To achieve this, we employ a comprehensive approach by characterizing the wealth and consumption distributions through three key statistics: mean, standard deviation, and percentiles. Our objective is to assess how changes in RRA and EIS impact these statistical measures, providing insights into the role of these parameters in shaping the stationary distribution within the model.

4.1 Wealth Distribution

We start by inspecting the effects of RRA (γ) and EIS (ψ) on the aggregate wealth distribution (see Table 2). First, independently of the level of γ and ψ, the mean of wealth is zero, suggesting that the economy is characterized by borrowers (agents with $a < 0$) and lenders (agents with $a > 0$). Second, an increase of ψ generates higher dispersion (standard deviation), which can be interpreted as more inequality. However, changes in γ have marginal or non-effects on dispersion. This is mainly explained from the different effects of γ and ψ on the interest rate.

Assessing the effects on wealth percentiles, we observe the following. A change in ψ decreases the first three percentiles (1%, 5%, and 25%), all of which are negative. Consequently, an increase in ψ elevates the level of debt in the economy. Simultaneously, an increase in ψ raises the last three percentiles (75%, 95%, and 99%)–which are positive, indicating that wealthy individuals become wealthier. Therefore, ψ has significant effects on wealth distribution, increasing its inequality, while the effects of γ are negligible.

Table 2: Wealth Distribution (total population)

ψ	Mean	Std. dev.	1%	5%	25%	50%	75%	95%	99%
(A) $\gamma = 3$									
0.2	0.000	0.044	-0.148	-0.105	-0.013	0.021	0.029	0.034	0.035
0.5	0.000	0.049	-0.149	-0.113	-0.021	0.021	0.037	0.044	0.045
(B) $\gamma = 5$									
0.2	0.000	0.043	-0.148	-0.103	-0.013	0.020	0.029	0.034	0.035
0.5	0.000	0.049	-0.149	-0.112	-0.020	0.020	0.036	0.043	0.045

We then explore the effects of RRA and EIS on distributions by agent type (see Table 3). First, regardless of the level of γ and ψ, the wealth mean of low-income agents is negative, while the same statistic is positive for high-income agents. This observation suggests that low-income agents are the net borrowers and high-income agents are the net lenders in this economy. Second, based on percentiles, 75% of low-income agents are borrowers, contrasting with the fact that only 25% of high-income agents are borrowers.

Similarly, as observed at the aggregate level, variations in γ have no effects or, at best, marginal effects on agent-type wealth distribution. However, this is not the case for changes in ψ. First, an increase in ψ leads to increased dispersion in the wealth distribution of both agent types, with strong effects on high-income agents. Second, the more significant effects of ψ on the wealth distribution of low-income agents are observed in the last four percentiles (50%, 75%, 95%, and 99%). Specifically, percentiles 50% and 75% become more negative, indicating that an increase in ψ elevates the debt level of these agents. Additionally, percentiles 95% and 98% become more positive, suggesting that low-income agents with high wealth become even wealthier. Third, in contrast with low-income agents, changes in ψ affect all wealth percentiles. Consequently, the 25% of these agents are now more leveraged, while the last four percentiles indicate that agents with high wealth become wealthier.

These results highlight that changes in ψ not only affect aggregate distributions but also have distinct impacts on agent-type wealth distribution. These heterogeneous effects of ψ appear to be economically significant and should be carefully considered in any heterogeneous-agent model.

Table 3: Wealth Distribution (agent types)

ψ	Mean	Std. dev.	1%	5%	25%	50%	75%	95%	99%
(1) Low-income agent									
(A) $\gamma = 3$									
0.2	-0.088	0.057	-0.150	-0.150	-0.146	-0.099	-0.040	0.014	0.028
0.5	-0.090	0.059	-0.150	-0.150	-0.147	-0.106	-0.042	0.018	0.036
(B) $\gamma = 5$									
0.2	-0.088	0.057	-0.150	-0.150	-0.145	-0.098	-0.039	0.014	0.028
0.5	-0.090	0.059	-0.150	-0.150	-0.146	-0.105	-0.042	0.018	0.035
(2) High-income agent									
(A) $\gamma = 3$									
0.2	0.005	0.038	-0.129	-0.083	-0.006	0.022	0.029	0.034	0.035
0.5	0.005	0.044	-0.134	-0.093	-0.014	0.023	0.037	0.044	0.045
(B) $\gamma = 5$									
0.2	0.005	0.037	-0.128	-0.082	-0.006	0.021	0.029	0.034	0.035
0.5	0.005	0.043	-0.134	-0.092	-0.013	0.023	0.036	0.043	0.045

4.2 Consumption Distribution

After examining the impacts of RRA and EIS on wealth distribution, we turn our attention to their effects on consumption distribution, beginning with an aggregate-level analysis (see Table 4). A key finding is that, at the aggregate level, changes in γ do not alter the standard deviation, whereas changes in ψ do. Specifically, an increase in ψ from 0.2 to 0.5 results in higher consumption dispersion.

This larger dispersion can be attributed to two factors: first, a reduction in consumption at the first percentile, primarily driven by a high concentration of low-income agents constrained by borrowing limitations ($a \geq \underline{a} \equiv -0.15$), who consequently experience elevated equilibrium interest rates leading to reduced consumption. Second, an increase in consumption at the higher percentiles (75%, 95%, and 99%), due to these agents being net lenders, profiting from the high interest rates and subsequently boosting their consumption.

Table 4: Consumption Distribution (total population)

ψ	Mean	Std. dev.	1%	5%	25%	50%	75%	95%	99%
(A) $\gamma = 3$									
0.2	1.000	0.0467	0.735	0.909	1.007	1.013	1.015	1.016	1.016
0.5	1.000	0.0473	0.727	0.921	1.006	1.013	1.016	1.017	1.017
(B) $\gamma = 5$									
0.2	1.000	0.0467	0.736	0.906	1.007	1.014	1.015	1.015	1.015
0.5	1.000	0.0473	0.728	0.919	1.006	1.013	1.016	1.017	1.017

What are the effects of RRA and EIS on consumption distribution by agent type? The results presented in Table 5 address this question. Firstly, it's important to note that regardless of the level of γ and ψ, the consumption dispersion of low-income agents remains lower than that of high-income agents. Secondly, an increase in ψ significantly boosts consumption dispersion for low-income agents, while changes in γ have only marginal effects on it. Thirdly, alterations in γ and ψ don't impact the consumption dispersion of high-income agents, although changes in ψ slightly shift their distribution to the right

Table 5: Consumption Distribution (agent types)

ψ	Mean	Std. dev.	1%	5%	25%	50%	75%	95%	99%
(1) Low-income agent									
(A) $\gamma = 3$									
0.2	0.810	0.066	0.718	0.718	0.746	0.823	0.870	0.899	0.906
0.5	0.810	0.073	0.714	0.714	0.739	0.824	0.877	0.910	0.918
(B) $\gamma = 5$									
0.2	0.810	0.065	0.718	0.718	0.748	0.823	0.868	0.897	0.904
0.5	0.810	0.072	0.714	0.714	0.740	0.823	0.875	0.908	0.916
(2) High-income agent									
(A) $\gamma = 3$									
0.2	1.010	0.008	0.979	0.992	1.009	1.014	1.015	1.016	1.016
0.5	1.010	0.008	0.981	0.991	1.007	1.014	1.016	1.017	1.017
(B) $\gamma = 5$									
0.2	1.010	0.008	0.979	0.992	1.009	1.014	1.015	1.016	1.016
0.5	1.010	0.008	0.981	0.991	1.008	1.014	1.016	1.017	1.017

5 Leverage

We devote this section to exploring how leverage is determined in the economy and how RRA and EIS affect it. Our analysis is confined to the set of values of (γ, ψ) for which the model has a solution (see Subsection 3.3).

Figure 6 depicts the relationship between leverage ratio and RRA, EIS, and interest rate. An interesting theoretical result emerges: given ψ, the leverage ratio decreases with RRA. This negative slope of the leverage ratio with respect to RRA is observable in the middle graph of Figure 6. What explains this slope? When agents exhibit greater risk aversion (an increase in γ), they seek to hedge against labor income risk to smooth consumption between different states of nature (low and high income states). Consequently, they demand more bonds, driving up bond prices and thereby reducing the interest rate. With the new, lower equilibrium interest rate, agents are discouraged from investing in bonds, leading to a reduction in total leverage in the economy.

Another important finding is that, given a specific RRA value (for example, $\gamma = 4$), an increase in EIS (ψ) leads to a monotonically increasing leverage ratio. This trend is primarily driven by the sensitivity of the interest rate to changes in ψ. It may seem surprising at first glance that the leverage ratio exhibits a positive slope with respect to the interest rate. However, this economic phenomenon can be attributed to the dominance of the income effect over the substitution effect, given that EIS is less than one. As a result, the partial derivative of consumption with respect to the interest rate $(\partial c_j(a)/\partial r)$ is positive (Achdou et al., 2022). Therefore, an increase in ψ leads to a rise in r, which in turn increases consumption while reducing saving, thereby resulting in an increase in leverage.

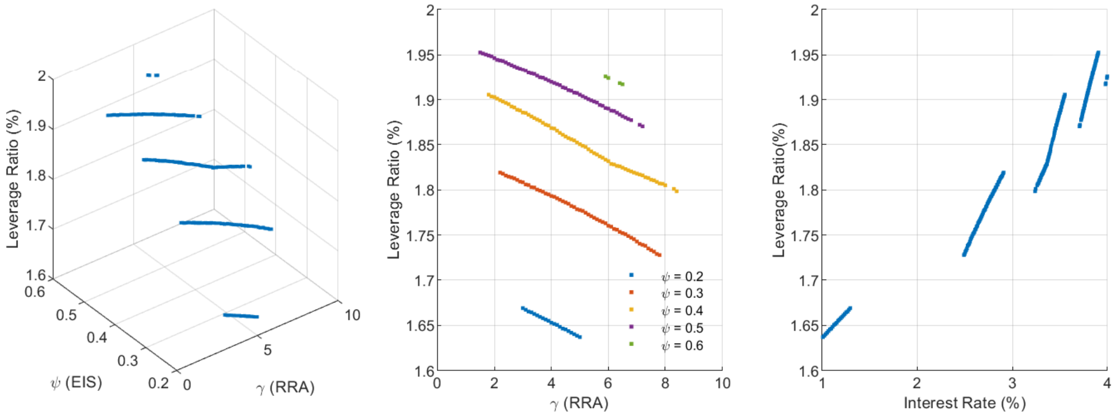

Figure 6: **Leverage Ratio.** This figure depicts the relationship between the leverage ratio and the Elasticity of Intertemporal Substitution (EIS), Relative Risk Aversion (RRA), and interest rate. The leverage ratio is calculated as the aggregate leverage over total income at the stationary equilibrium. Aggregate leverage is defined as the sum of $(g_1 + g_2)a$ for $a < 0$, where g_j for $j \in \{1, 2\}$ represent the joint density of income y_j and wealth a. Total income is calculated as the sum of $(g_1y_1 + g_2y_2)$ for all values of a within the range $[-0.15, 5]$, where y_1 and y_2 represent the low and high income, respectively.

6 Conclusions

We extend the Huggett model to incorporate Epstein and Zin (1989) preferences in continuous time and analyze the roles of Relative Risk Aversion (RRA) and Elasticity of Intertemporal Substitution (EIS) in determining equilibrium interest rates, consumption-saving decisions, stationary distributions, and leverage.

Our findings suggest that variations in the EIS have stronger effects on equilibrium outcomes than variations in RRA. Furthermore, we demonstrate that equilibrium exists only within a subset of values for RRA and EIS, namely when $RRA > 1$ and $EIS < 1$. Therefore, gaining an understanding of structural parameters in heterogeneous-agent models requires a quantitative distinction between RRA and EIS.

These results carry significant implications for macroeconomic and asset pricing models. Firstly, long-run risk models typically assume $EIS > 1$. However, our findings raise an important question: Can our model, enhanced with aggregate shocks, effectively address asset return puzzles when EIS is less than one? This presents a promising avenue for future research. Secondly, understanding the distributional effects of state variables or financial market variables necessitates distinguishing between both parameters, RRA and EIS, as they exert differing impacts at both the aggregate and agent-level distributions.

Overall, our findings underscore the importance of quantitatively distinguishing between RRA and EIS in heterogeneous-agent models.

Appendix

A Lemma Proofs

A.1 Proof of Lemma 2.1.

The strategy: First, we start with the optimization problem in discrete time with length period Δ. Second, we take the limit of the HJB equation with respect to $\Delta \to 0$. This last step allows us to obtain the HJB equation in continuous time.

(a) Discrete-time problem. We first set a length period equals to Δ. Second, the agent with income y_j in period t keeps their income in period $t+\Delta$ with probability $P_j(\Delta) = e^{-\lambda_j \Delta}$ and switch to state y_{-j} with probability $1 - P_j(\Delta)$. Then, the Bellman equation for this problem is given by

$$V_j(a) = \max_{\{c_t\}} \left\{ f(c_j, V_j)\Delta + \beta(\Delta) \underbrace{\left[P_j(\Delta)V_j(a_{t+\Delta}) + (1 - P_j(\Delta))V_{-j}(a_{t+\Delta})\right]}_{E[V_j(a_{t+\Delta})]} \right\} \quad (34)$$

subject to

$$a_{t+\Delta} = (y_{jt} + ra_t - c_t)\,\Delta + a_t, \quad (35)$$

$$a_{t+\Delta} \geq \underline{a}, \quad (36)$$

for $j \in \{1, 2\}$.

Importantly, the expected value function of agent j ($E\left[V_j(a_{t+\Delta})\right]$) considers the possibility to have the same income y_j and to jump to the other income y_{-j} in the next period $t + \Delta$:

$$E[V_j(a_{t+\Delta})] = P_j(\Delta)V_j(a_{t+\Delta}) + (1 - P_j(\Delta))V_{-j}(a_{t+\Delta})$$

(b) Taking limit when $\Delta \to 0$. We then take limits on the conditional probabilities, and then on the HJB equation when the time length tends to zero ($\Delta \to 0$).

$$P_j(\Delta) = e^{-\lambda_j \Delta} \Rightarrow P_j(\Delta) \approx 1 - \lambda_j \Delta \quad (37)$$

In the HJB equation (34)

$$V_j(a_t) = \max_{\{c\}} \{f(c_j, V_j)\Delta + [(1 - \lambda_j\Delta)V_j(a_{t+\Delta}) + \lambda_j \Delta V_{-j}(a_{t+\Delta})]\}$$

$$0 = \max_{\{c\}} \{f(c_j, V_j))\Delta + [(1 - \lambda_j\Delta)V_j(a_{t+\Delta}) + \lambda_j \Delta V_{-j}(a_{t+\Delta}) - V_j(a_t)]\}$$

$$0 = \max_{\{c\}} \{f(c_j, V_j)\Delta + [V_j(a_{t+\Delta}) - V_j(a_t) + \lambda_j \Delta[V_{-j}(a_{t+\Delta}) - V_j(a_{t+\Delta})]]\} \quad (38)$$

Dividing the Eq. (38) by Δ, it turns out

$$0 = \max_{\{c\}} \left\{ \frac{f(c_j, V_j)\Delta}{\Delta} + \left[\frac{V_j(a_{t+\Delta}) - V_j(a_t)}{\Delta} + \lambda_j (V_{-j}(a_{t+\Delta}) - V_j(a_{t+\Delta})) \right] \right\}$$

Taking limit when $\Delta \to 0$

$$0 = \max_{\{c\}} \left\{ f(c_j, V_j) + \lim_{\Delta \to 0} \left[\frac{V_j(a_{t+\Delta}) - V_j(a_t)}{\Delta} \right] + \lambda_j (V_{-j}(a_t) - V_j(a_t)) \right\} \quad (39)$$

It is worth noting that the value function depends on the state variable a and indirectly on the time. Then, the derivative of the value function is respect to its variable a and not respect to the time t. The term in "lim" operator, in the Eq. (39), has a denominator Δ: a change in time. We need to change this denominator to have "change in a." With this goal in mind, we can use the expression for $a_{t+\Delta}$ from the Eq. (35) to rewrite the limit term, as follows

$$\begin{aligned}
\lim_{\Delta \to 0} \left[\frac{V_j(a_{t+\Delta}) - V_j(a_t)}{\Delta} \right] &= \lim_{\Delta \to 0} \left[\frac{V_j(a_t + \Delta(y_{jt} + r_t a_t - c_t)) - V_j(a_t)}{\Delta} \right] \\
&= \lim_{\Delta \to 0} \left[\frac{V_j(a_t + \Delta(y_{jt} + r_t a_t - c_t)) - V_j(a_t)}{\Delta(y_{jt} + r_t a_t + c_t)} \frac{\Delta(y_{jt} + r_t a_t + c_t)}{\Delta} \right] \\
&= \lim_{x \to 0} \left[\frac{V_j(a_t + x) - V_j(a_t)}{x} \frac{(y_{jt} + r_t a_t + c_t)}{1} \right] \\
&= V_j'(a_t)(y_{jt} + r_t a_t + c_t), \quad (40)
\end{aligned}$$

where $x \triangleq \Delta(y_{jt} + r_t a_t - c_t)$. Then, $\Delta \to 0$ is equivalent to $x \to 0$. Considering Eq. (40) into Eq. (39), **the HJB** equation becomes

$$0 = \max_{\{c\}} \left\{ f(c_j, V_j) + V_j'(a)(y_j + r_t a - c) + \lambda_j (V_{-j}(a) - V_j(a)) \right\} \quad (41)$$

with the law of movement of the state variable (agent's saving) (42) and the borrowing constraint (43), which in continuous-time are defied as

$$\begin{aligned}
a_{t+\Delta} &= \Delta(y_{jt} + r_t a_t - c_t) + a_t \\
\frac{a_{t+\Delta} - a_t}{\Delta} &= y_{jt} + r_t a_t - c_t \\
\lim_{\Delta \to 0} \left[\frac{a_{t+\Delta} - a_t}{\Delta} \right] \equiv \dot{a}_t &= y_{jt} + r_t a_t - c_t \quad (42) \\
a_{t+\Delta} \geq \underline{a} &\to \lim_{\Delta \to 0} : a_t \geq \underline{a} \quad (43)
\end{aligned}$$

Remark. We know that the wealth $a \in [\underline{a}, \infty^+[$. Then, in the interior of the state space, we have $a_t > \underline{a}$. Now, we define Δ arbitrarily small. Therefore, $a_t > \underline{a}$ implies $a_{t+\Delta} > \underline{a}$.

$$\lim_{\Delta \to 0} a_{t+\Delta} = a_t \text{ and we know } a_t > \underline{a} \quad (44)$$

Then, if $a_t > \underline{a} \Rightarrow$ for Δ small $\Rightarrow a_{t+\Delta} > \underline{a}$. This fact implies that the borrowing constraint $a_t > \underline{a}$ never binds in the interior of the state space.

A.2 Proof of Lemma 2.2.

The next step is to derive a law of movement of the distribution of the state variable a. We follow the same strategy that we applied to the HJB equation. First, we start with a discrete-time approach assuming a length of period Δ small. Then, we move to a continuous-time environment by taking the limit when Δ tends to zero.

A.2.1 Preliminary

In discrete-time economy, we define the following. First, $G_j(a,t)$ represents the fraction of population with income y_j and wealth below a level in period t:

$$G_j(a,t) = \text{Prob}(\tilde{a}_t \leq a, \tilde{y}_t = y_j), \tag{45}$$

where $\tilde{a}_t$ and $\tilde{y}_t$ represent agent's wealth and income as a random variables, while a and y_j represent a *level* of these variables. Second, we evaluate $G_j(a,t)$ at the borrowing limit $\underline{a}$, as follows.

$$G_1(\underline{a},t) + G_2(\underline{a},t) = 0, \quad \forall t, \tag{46}$$

where,

- $G_1(\underline{a},t)$ is the fraction of people with income y_1 and wealth lower or equals to "a"
- $G_2(\underline{a},t)$ is the fraction of people with income y_2 and wealth lower or equals to "a"

Then,

$$G_1(\underline{a},t) = G_2(\underline{a},t) = 0 \tag{47}$$

Finally, the total population is normalized to one as follows.

$$\lim_{a\to\infty^+}[G_1(a,t) + G_2(a,t)] = 1, \tag{48}$$

where,

- $G_1(a,t)$: The function of people with income y_1, "$\lim_{a\to\infty} G_1(a,t)$"
- $G_2(a,t)$: The function of people with income y_2, "$\lim_{a\to\infty} G_2(a,t)$"

A.2.2 Law of motion for $G_j(a,t)$ in discrete time

We want to derive a *law of motion for* G_j; i.e., we are interested in *how* G_j *changes over time*. Specifically, we want to calculate the following:

$$\frac{\partial G_j(a,t)}{\partial t} = \lim_{\Delta\to 0}\left[\frac{G_j(a,t+\Delta) - G_j(a,t)}{\Delta}\right], \quad j \in \{1,2\} \tag{49}$$

To define the Eq. (49), we need to find an expression for $G_j(a,t+\Delta)$. Therefore, our first goal is to find that expression. To do so, we recall the definition of $G_j(a,t+\Delta)$:

$$G_j(a, t + \Delta) = \text{Prob}[\tilde{a}_{t+\Delta} \leq a, \tilde{y}_{t+\Delta} = y_j]. \tag{50}$$

To find an expression of Eq. (50), we proceed in two steps. First, we compute $\text{Prob}[\tilde{a}_{t+\Delta} \leq a]$ without considering income change between t and $t + \Delta$. Second, we consider the change in income. We then use the $G_j(a, t + \Delta)$ obtained in the previous steps to calculate $\partial G_j(a, t)/\partial t$ (the Eq. 49), which is the law of motion for G_j.

Additionally, to find the law of motion for G_j, we need to answer the following question: "if a type j individual has wealth $a_{t+\Delta}$ at time $t + \Delta$, then what level of wealth $\tilde{a}_t$ did he have at period t?" We know the law of movement of the state variable $\tilde{a}$ as a definition of saving:

$$\tilde{a}_{t+\Delta} = \tilde{a}_t + \Delta s_j(\tilde{a}_t) \tag{51}$$
$$\tilde{a}_{t+\Delta} = \tilde{a}_t + \Delta s_j(\tilde{a}_{t+\Delta}) \tag{52}$$

We use the equation (52) because it is convenient. Importantly, both equation (51 and 52) are the same; the difference between them is that the former looks forward in time and the latter looks backward. Since the HJB equation is forward looking, we will use Eq. (52).

$$\tilde{a}_t = \tilde{a}_{t+\Delta} - \Delta s_j(\tilde{a}_{t+\Delta}) \tag{53}$$

Intuition: If $s_j(\tilde{a}_{t+\Delta}) < 0$ (this means the agent dissaves), his past wealth $\tilde{a}_t$ must have been larger than his current wealth $\tilde{a}_{t+\Delta}$.

We now calculate $\text{Prob}[\tilde{a}_{t+\Delta} \leq a]$ in two steps.

Step 1 (labor income does not change). We analyze the wealth in $t + \Delta$ without considering the change in income. Fig. 7 illustrates the movement of the fraction of people from t to $t + \Delta$ that have wealth below a in $t + \Delta$. Specifically, under the assumption of dissaving, $s_j \leq 0$, Fig. 7 shows that the probability of wealth in $t + \Delta$ to be below than a (i.e., $\text{Prob}[\tilde{a}_{t+\Delta} \leq a]$) comes from two sources:

- First, the fraction of population X that already had wealth below a in t. Since they dissaves, their wealth in $t + \Delta$ would be lower than a.

$$X = \text{Prob}[\tilde{a}_t \leq a] \tag{54}$$

- Second, the fraction of population Y that had wealth higher than a in t, but since they dissaved in t, their wealth in $t + \Delta$ would be below than a for some threshold a_t^*.

$$Y = \text{Prob}[a \leq \tilde{a}_t \leq a_t^*] \tag{55}$$

We need to calculate the level of wealth in t that allows us to get a in $t+\Delta$: $\tilde{a}_{t+\Delta} = a$. Then, using the Eq. (53), we can obtain a_t^*:

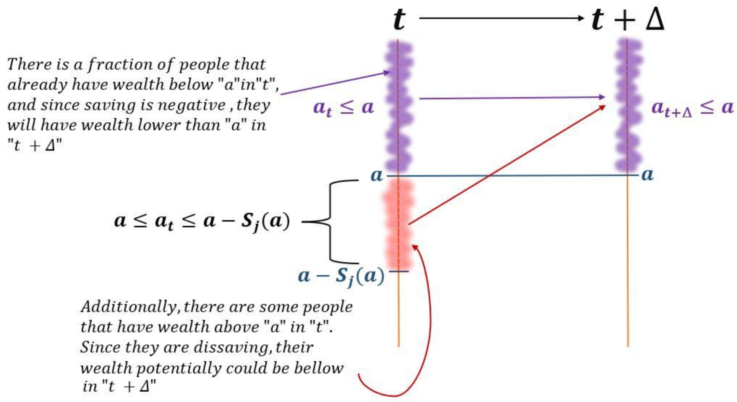

Figure 7: Transition of wealth

$$
\begin{aligned}
\tilde{a}_t &= \tilde{a}_{t+\Delta} - \Delta s_j(\tilde{a}_{t+\Delta}) \\
a_t^* &= a - \Delta s_j(a)
\end{aligned}
\tag{56}
$$

Introducing a_t^* from Eq. (56) into Eq. (55):

$$
Y = \text{Prob}[a \leq \tilde{a}_t \leq a - \Delta s_j(a)] \tag{57}
$$

With these two sources, we now can calculate $\text{Prob}[\tilde{a}_{t+\Delta} \leq a]$:

$$
\begin{aligned}
\text{Prob}[\tilde{a}_{t+\Delta} \leq a] &= X + Y \\
&= \text{Prob}[\tilde{a}_t \leq a] + \text{Prob}[a \leq \tilde{a}_t \leq a - \Delta s_j(a)] \\
\text{Prob}[\tilde{a}_{t+\Delta} \leq a] &= \text{Prob}[\tilde{a}_t \leq a - \Delta s_j(a)]
\end{aligned}
\tag{58}
$$

Recall, this probability (Eq. 58) does not consider the change in income between t and $t + \Delta$. The next step is to consider the transition of income.

Step 2 (labor income changes). Now, we consider the possibility that some people from t with income y_j and income y_{-j} could have income y_j in period $t + \Delta$. Fig. 8 illustrates the changes in income and their probabilities. In particular, to calculate the probability of the wealth in $t + \Delta$ to be below a given that the income in that period is y_1, we need to take into account two sources:

- First, the fraction of the population that had income y_1 in t and have the same income in $t + \Delta$. The probability to have the same y_1 income is $(1 - \lambda_1 \Delta)$:

$$
Pr[y_{t+\Delta} = y_1 | y_t = y_1] = 1 - \lambda_1 \Delta
$$

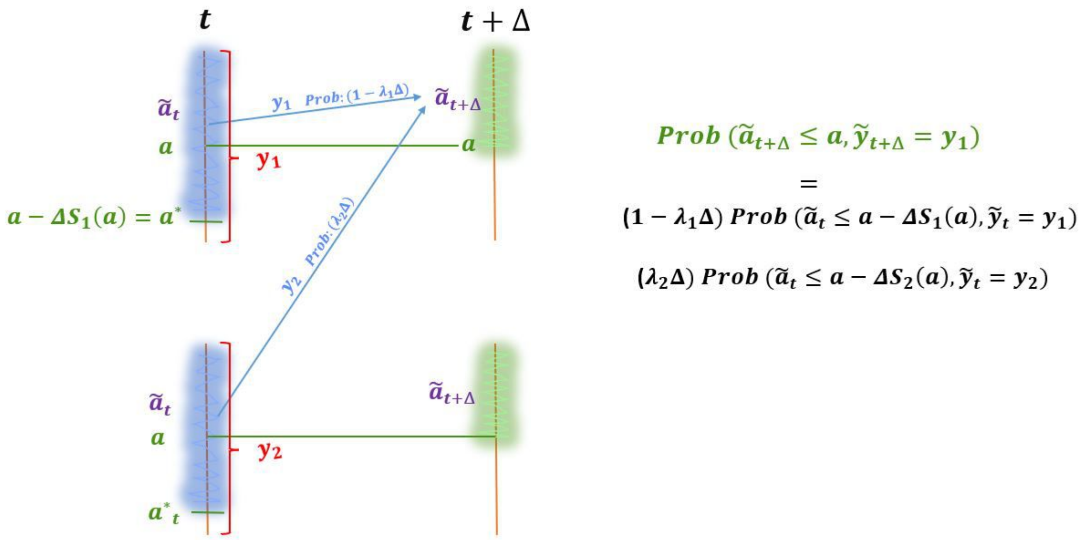

Figure 8: Transition of wealth with income

- Second, the fraction of the population that had income y_2 and now, in $t+\Delta$, they have income y_1. The probability to pass from income y_2 in t to y_1 in $t+\Delta$ is $(\lambda_2\Delta)$:

$$Pr[y_{t+\Delta} = y_1 | y_t = y_2] = \lambda_2\Delta$$

Taking into account these two sources, the probability of the wealth to be below a in $t+\Delta$ for income y_1 is:

$$\begin{aligned} \text{Prob}[\tilde{a}_{t+\Delta} \leq a, \tilde{y}_{t+\Delta} = y_1] &= Prob[y_{t+\Delta} = y_1 | y_t = y_1]\text{Prob}[\tilde{a}_t \leq a - \Delta S_1(a), \tilde{y}_t = y_1] \\ &+ Prob[y_{t+\Delta} = y_1 | y_t = y_2]\text{Prob}[\tilde{a}_t \leq a - \Delta S_2(a), \tilde{y}_t = y_2] \\ \text{Prob}[\tilde{a}_{t+\Delta} \leq a, \tilde{y}_{t+\Delta} = y_1] &= (1-\lambda_1\Delta)G_1(a - \Delta S_1(a), t) + (\lambda_2\Delta)G_2(a - \Delta S_2(a), t) \end{aligned}$$

In general terms, for any income y_j:

$$\underbrace{\text{Prob}[\tilde{a}_{t+\Delta} \leq a, \tilde{y}_{t+\Delta} = y_j]}_{G_j(a,t+\Delta)} = (1-\lambda_j\Delta)G_j(a-\Delta s_j(a), t) + (\lambda_{-j}\Delta)G_{-j}(a-\Delta s_{-j}(a), t) \quad (59)$$

Calculating the law of motion of $G_j(a,t)$. Since we have an expression of $G_j(a, t+\Delta)$ (Eq. 59), we can calculate $\partial G_j(a,t)/\partial t$ (the law of motion for G_j stated in Eq. 49). to do so, we start introducing $-G_j(a,t)$ in both side of Eq. (59), as follows.

$$\begin{aligned} G_j(a, t+\Delta) - G_j(a,t) &= (1-\lambda_j\Delta)G_j(a-\Delta s_j(a), t) - G_j(a,t) + (\lambda_{-j}\Delta)G_{-j}(a-\Delta s_{-j}(a), t) \\ G_j(a, t+\Delta) - G_j(a,t) &= G_j(a-\Delta s_j(a), t) - G_j(a,t) \\ &- (\lambda_j\Delta)G_j(a-\Delta s_j(a), t) + (\lambda_{-j}\Delta)G_{-j}(a-\Delta s_{-j}(a), t) \quad (60) \end{aligned}$$

Dividing Eq. (60) by Δ:

$$\begin{aligned}\frac{G_j(a,t+\Delta)-G_j(a,t)}{\Delta} &= \frac{G_j(a-\Delta s_j(a),t)-G_j(a,t)}{\Delta} \\ &- \left[\frac{(\lambda_j\Delta)G_j(a-\Delta s_j(a),t)-(\lambda_{-j}\Delta)G_{-j}(a-\Delta s_{-j}(a),t)}{\Delta}\right]\end{aligned} \tag{61}$$

Δ in the last term is ruled out from the numerator and denominator. As a result, Eq. (61) turns out:

$$\begin{aligned}\frac{G_j(a,t+\Delta)-G_j(a,t)}{\Delta} &= \frac{G_j(a-\Delta s_j(a),t)-G_j(a,t)}{\Delta} \\ &- (\lambda_j)G_j(a-\Delta s_j(a),t)+(\lambda_{-j})G_{-j}(a-\Delta s_{-j}(a),t)\end{aligned} \tag{62}$$

To simplify the algebra, we express the Eq. (62) in three terms:

$$A = B + C, \tag{63}$$

where,

$$\begin{aligned}A &= \frac{G_j(a,t+\Delta)-G_j(a,t)}{\Delta} \\ B &= \frac{G_j(a-\Delta s_j(a),t)-G_j(a,t)}{\Delta} \\ C &= -(\lambda_j)G_j(a-\Delta s_j(a),t)+(\lambda_{-j})G_{-j}(a-\Delta s_{-j}(a),t)\end{aligned}$$

A.2.3 Law of motion for $G_j(a,t)$ in continuous time

Next, we take $\lim_{\Delta\to 0}$ to the Eq. (63).

$$\lim_{\Delta\to 0} A = \lim_{\Delta\to 0} B + \lim_{\Delta\to 0} C$$

First term:

$$\lim_{\Delta\to 0} A = \lim_{\Delta\to 0}\left[\frac{G_j(a,t+\Delta)-G_j(a,t)}{\Delta}\right] = \partial_t G_j(a,t)$$

Second term:

$$\begin{aligned}\lim_{\Delta\to 0} B &= \lim_{\Delta\to 0}\left[\frac{G_j(a-\Delta s_j(a),t)-G_j(a,t)}{\Delta}\right] \\ &= \lim_{\Delta\to 0}\left\{\left[\frac{G_j(a-\Delta s_j(a),t)-G_j(a,t)}{-\Delta s_j(a)}\right]\left(\frac{-\Delta s_j(a)}{\Delta}\right)\right\} \\ &= \lim_{\Delta\to 0}\left\{\left[\frac{G_j(a-\Delta s_j(a),t)-G_j(a,t)}{-\Delta s_j(a)}\right](-s_j(a))\right\} \\ &= \lim_{\Delta\to 0}\left\{\left[\frac{G_j(a-\Delta s_j(a),t)-G_j(a,t)}{-\Delta s_j(a)}\right]\right\}(-s_j(a)) \\ &= \partial_a G_j(a,t)(-s_j(a))\end{aligned}$$

Third term:

$$
\begin{aligned}
\lim_{\Delta\to 0} C &= \lim_{\Delta\to 0}\left[-(\lambda_j)G_j(a-\Delta s_j(a),t)+(\lambda_{-j})G_{-j}(a-\Delta s_{-j}(a),t)\right] \\
&= -(\lambda_j)\lim_{\Delta\to 0}[G_j(a-\Delta s_j(a),t)]+(\lambda_{-j})\lim_{\Delta\to 0}[G_{-j}(a-\Delta s_{-j}(a),t)] \\
&= -(\lambda_j)[G_j(a,t)]+(\lambda_{-j})[G_{-j}(a,t)] \qquad (64)
\end{aligned}
$$

Therefore, using the limit of A, B, and C, the Eq. (63) turns out:

$$
\partial_t G_j(a,t) = -s_j(a)[\partial_a G_j(a,t)] - \lambda_j G_j(a,t) + \lambda_{-j} G_{-j}(a,t) \qquad (65)
$$

We proceed with two more steps. First, we know a relationship between the CDF $G_j(a,t)$ and the density function $g_j(a,t)$:

$$
\partial_a G_j(a,t) = g_j(a,t)
$$

Using this relationship in the Eq. (65):

$$
\partial_t G_j(a,t) = -s_j(a) g_j(a,t) - \lambda_j G_j(a,t) + \lambda_{-j} G_{-j}(a,t) \qquad (66)
$$

Second, we derive the Eq. (66) with respect to "a":

$$
\begin{aligned}
\partial_t \underbrace{\partial_a G_j(a,t)}_{g_j(a,t)} &= -\partial_a[s_j(a)g_j(a,t)] - \lambda_j g_j(a,t) + \lambda_{-j} g_{-j}(a,t) \\
\partial_t g_j(a,t) &= -\partial_a[s_j(a)g_j(a,t)] - \lambda_j g_j(a,t) + \lambda_{-j} g_{-j}(a,t) \qquad (67) \\
0 &= -\partial_a[s_j(a)g_j(a)] - \lambda_j g_j(a) + \lambda_{-j} g_{-j}(a) \qquad (68)
\end{aligned}
$$

The expression (67) is the Kolmogorov Forward equation over time for agent j. This partial differential equation captures the movement of the distribution $g_j(a,t)$ over time. To compute the *stationary distribution*, $g_j(a,t)$ keeps constant over time. We can obtain this stationary distribution considering $\partial_t g_j(a,t) = 0$, which is Eq. (68).

B The Numerical Solution Method

In this appendix, I provide details of the solution method.

B.1 The Mathematical Model: The HJB Equation

The mathematical model is represented by a system of PDEs formed by the PDE of the value function V_j for $j \in \{1, 2\}$. The PDE equation for agent j (Eq. 25) is given by

$$0 = f(c_j, V_j) + V_j'(a)\,(y_j + ra - c_j) + \lambda_j\,(V_{-j}(a) - V_j(a))\,. \tag{69}$$

Considering the full expression of $f(c_j, V_j)$, from Eq. (33), into Eq. (69), it becomes

$$0 = \frac{V_j}{\theta}\left[c_j^{1-\delta}\,[(1-\gamma)V_j]^{-\theta} - \rho\right] + V_j'(a)\,(y_j + ra - c_j) + \lambda_j\,(V_{-j}(a) - V_j(a))\,, \tag{70}$$

Since $\theta \triangleq (1-\delta)/(1-\gamma)$ could be positive or negative, it should be consider into the finite difference approximation of the first derivative. To do so, we multiply the Equation (70) by θ to consider the effect of its sign on the finite difference method. Note that θ affects the coefficient of $V_j'(a)$, thereby influencing the decision on whether to use forward or backward difference approximation. After multiplying Equation (70) by θ, it becomes:

$$0 = V_j\left[c_j^{1-\delta}\,[(1-\gamma)V_j]^{-\theta} - \rho\right] + V_j'(a)\theta\,(y_j + ra - c_j) + \lambda_j\theta\,(V_{-j}(a) - V_j(a))\,. \tag{71}$$

This PDE is accompanied by the FOC, and the dynamics of the state variable (wealth):

$$((1-\gamma)V_j)^{1-\theta}\,c_j^{-\delta} = V_j'(a) \tag{72}$$

$$da = (y_j + ra - c_j)\,dt \tag{73}$$

Since savings s_j is defined as $s_j = da/dt$, Eq. (73) can be expressed as

$$s_j = y_j + ra - c_j$$

We use this expression instead of (73).

B.2 State Space and PDE Discretization

We use a structured grid for the state space–an equispaced wealth a grid. Then, I evaluate each PDE at every point of the grid, resulting in the *discretized* PDEs system for $j \in \{1, 2\}$, which is given by

$$\frac{V_{i,j}^{n+1} - V_{i,j}^{n}}{\Delta} = V_{i,j}\left[c_{i,j}^{1-\delta}\,[(1-\gamma)V_{i,j}]^{-\theta} - \rho\right] + V_{i,j}'\theta\,(y_j + ra_i - c_{i,j}) + \lambda_j\theta\,(V_{i,-j} - V_{i,j})\,, \tag{74}$$

in which we add $\left(V_{i,j}^{n+1} - V_{i,j}^{n}\right)/\Delta$ in the left-side of Eq. (74) to consider the value function iteration in the following steps. This discretized PDE is accompanied by

$$c_{i,j} = \left(V'_{i,j}\left((1-\gamma)V_{i,j}\right)^{\theta-1}\right)^{-1/\delta} \tag{75}$$

$$s_{i,j} = y_j + ra_i - c_{i,j} \tag{76}$$

where $V'_{i,j}$ is either the *forward* or the *backward* difference approximation when the state variable a takes the value of a_i for $i = 1, \dots, I+1$.

B.3 Finite Difference

Since the PDE of V_j (Eq. 74) for agent type $j \in \{1, 2\}$ contains the first derivative of V_j, we define the forward and backward difference approximation of the first derivative of V_j as follows.

$$V'_{i,j} \approx \frac{V_{i+1,j} - V_{i,j}}{\Delta a} \equiv (V_{ij,F})' \quad : \text{Forward difference approximation} \tag{77}$$

$$V'_{i,j} \approx \frac{V_{i,j} - V_{i-1,j}}{\Delta a} \equiv (V_{ij,B})' \quad : \text{Backward difference approximation} \tag{78}$$

B.4 Upwind Scheme

The next step is to define when forward or backward approximation should be used. The criterion is provided by the Upwind scheme. I first define the coefficient of $V'_{i,j}$ as

$$\tilde{a}_{ij} = \theta\left(y_j + ra_i - c_{i,j}\right) \equiv \theta s_{i,j}, \qquad \theta \neq 0 \tag{79}$$

The upwind scheme suggests the following rule:

- Use **forward** approximation if the coefficient associated to $V'_{i,j}$ in the right side of the HJB equation is **positive**.
- Use **backward** approximation if the coefficient associated to $V'_{i,j}$ in the right side of the HJB equation is **negative**.

Therefore, this rule is given by

$$\tilde{a}_{ij} > 0 \longrightarrow V'_{i,j} \approx (V_{ij,F})', \quad \tilde{a}_{ij} = \tilde{a}_{ij,F} \tag{80}$$

$$\tilde{a}_{ij} < 0 \longrightarrow V'_{i,j} \approx (V_{ij,B})', \quad \tilde{a}_{ij} = \tilde{a}_{ij,B} \tag{81}$$

$$\tilde{a}_{ij} = 0 \longrightarrow V'_{i,j} \approx \left((1-\gamma)V_{i,j}\right)^{1-\theta}\left(y_j + ra_i\right)^{-\delta} \equiv (\overline{V}_{ij})'. \tag{82}$$

For the last case, we use the FOC (75) and the saving equation (76) using $s_{i,j} = 0$ (since $\theta \neq 0$). Furthermore, because $\tilde{a}_{ij}$ depends on $V'_{i,j}$ through $c_{i,j}$ (see FOC), it could be forward or backward. I then consider this fact in the coefficient $\tilde{a}_{ij}$, which is given by

$$\tilde{a}_{ij,F} = \theta s_{i,j,F} \equiv \theta\left(y_j + ra_i - c_{i,j,F}\right), \tag{83}$$

$$\tilde{a}_{ij,B} = \theta s_{i,j,B} \equiv \theta\left(y_j + ra_i - c_{i,j,B}\right). \tag{84}$$

The three cases stated in the rule of the Upwind scheme about the first derivative of V_j with respect to a (expressions 80 - 82) can be summarized in one equation.

$$V'_{i,j} = (V_{ij,F})' \mathbf{1}_{\tilde{a}_{ij,F}>0} + (V_{ij,B})' \mathbf{1}_{\tilde{a}_{ij,B}<0} + (\overline{V}_{ij})' \mathbf{1}_{\tilde{a}_{ij,F}<0<\tilde{a}_{ij,B}}, \tag{85}$$

where $\mathbf{1}_{\{\cdot\}}$ denotes an indicator function. The equation (85) represents the Upwind scheme indicating when we should use the forward or backward difference approximation of the first derivative of V_j w.r.t a. It also says what approximation to use when $\tilde{a}_{ij}$ is zero. Therefore, the discretized PDE of V_j is given by

$$\begin{aligned} \frac{V^{n+1}_{i,j} - V^{n}_{i,j}}{\Delta} &= V_{i,j} \left[c^{1-\delta}_{i,j} \left[(1-\gamma) V_{i,j} \right]^{-\theta} - \rho \right] \\ &+ \tilde{a}_{ij,F} (V_{ij,F})' + \tilde{a}_{ij,B} (V_{ij,B})' \\ &+ \lambda_j \theta \left(V_{i,-j} - V_{i,j} \right) \end{aligned} \tag{86}$$

with finite difference approximations (Eq. 80-82), the FOC (75), and the Upwind scheme rule (85).

B.5 Solution Method

Up to this point, we have the discretized PDE of V_j with the Finite Difference method and the Upwind scheme. Our next step is to set up the solution method (explicit or the implicit method). I use the implicit method for its outstanding properties in convergency (Candler, 2001; Achdou et al., 2022). After that, we express the system of $2 \times (I+1)$ equations from (87) in a matrix system, where $I+1$ is the number of grid points and the number *two* that multiplies $(I+1)$ reflects the fact that we have two agent types ($j \in \{1, 2\}$).

B.5.1 The Implicit Method

Using the implicit method, the discretized equation (87) is now evaluated in $n+1$.

$$\begin{aligned} \frac{V^{n+1}_{i,j} - V^{n}_{i,j}}{\Delta} &= V^{n}_{i,j} \left(c^{n}_{i,j} \right)^{1-\delta} \left[(1-\gamma) V^{n}_{i,j} \right]^{-\theta} - \rho V^{n+1}_{i,j} \\ &+ \tilde{a}^{n}_{ij,F} (V^{n+1}_{ij,F})' + \tilde{a}^{n}_{ij,B} (V^{n+1}_{ij,B})' \\ &+ \lambda_j \theta \left(V^{n+1}_{i,-j} - V^{n+1}_{i,j} \right) \end{aligned} \tag{87}$$

The implementation of the implicit method deserves some comments.

1. Three alternatives.

    - **Alternative 1.** Initially, we can think to use $f(c^{n}_{i,j}, V^{n}_{i,j})$ in (87). However, this is problematic since the term associated to ρ is used as an input and not as a variable. Convergence has problem in this case.

- **Alternative 2.** Working on $f(c_{i,j}, V_{i,j})$:

$$\begin{aligned} f(c_{i,j}, V_{i,j}) &= \frac{1}{1-\delta}(1-\gamma)V_{i,j} \times \left[c_{i,j}^{1-\delta}\left[(1-\gamma)V_{i,j}\right]^{-\frac{1-\delta}{1-\gamma}} - \rho\right] \\ &= \frac{V_{i,j}}{\theta} \times c_{i,j}^{1-\delta}\left[(1-\gamma)V_{i,j}\right]^{-\theta} - \rho\frac{V_{i,j}}{\theta} \\ &= \frac{V_{i,j}^{n}}{\theta} \times \left(c_{i,j}^{n}\right)^{1-\delta}\left[(1-\gamma)V_{i,j}^{n}\right]^{-\theta} - \rho\frac{V_{i,j}^{n+1}}{\theta} \\ &= f(c_{i,j}^{n}, V_{i,j}^{n}, V_{i,j}^{n+1}) \end{aligned} \tag{88}$$

We used Eq. (88) instead of $f(c_{i,j}^{n}, V_{i,j}^{n})$. Our argument is that if we use this approach for CRRA preferences ($\theta = 1$), Eq. (88) becomes

$$f_j(c_j, V_j) = \frac{\left(c_j^{n}\right)^{1-\delta}}{(1-\gamma)} - \rho V_j^{n+1},$$

which is the correct specification for CRRA. Then, this approach is consistent with CRRA when we shut down Epstein-Zin preference parameters.

- **Alternative 3.** Based on Candler (2001), we can do the following:

$$\begin{aligned} f(c_{i,j}, V_{i,j}) &= \frac{1}{1-\delta}(1-\gamma)V_{i,j} \times \left[c_{i,j}^{1-\delta}\left[(1-\gamma)V_{i,j}\right]^{-\frac{1-\delta}{1-\gamma}} - \rho\right] \\ &= \frac{V_{i,j}^{n+1}}{\theta} \times \left[\left(c_{i,j}^{n}\right)^{1-\delta}\left[(1-\gamma)V_{i,j}^{n}\right]^{-\theta} - \rho\right] \\ &= f(c_{i,j}^{n}, V_{i,j}^{n}, V_{i,j}^{n+1}) \end{aligned}$$

However, it does not work since $V_{i,j}^{n}$ and $V_{i,j}^{n+1}$ are multiplicative each other, making the system non-linear or we cannot separate $V_{i,j}^{n}$ and $V_{i,j}^{n+1}$.

2. The variables in the system (87) are the terms of $V_{i,j}^{n+1}$ for $i = 1 + I$. Then, to make this system linear, the coefficients that depend on $V_{i,j}$ should be evaluated in n. This allows the coefficients to be known.

3. The same approach is used in the coefficient of $(V_{ij,F}^{n+1})'$ and $(V_{ij,B}^{n+1})'$. These coefficients are evaluated at n as follows.

$$\tilde{a}_{ij,F}^{n} = \theta\left(y_j + ra_i - c_{i,j,F}^{n}\right), \tag{89}$$

$$\tilde{a}_{ij,B}^{n} = \theta\left(y_j + ra_i - c_{i,j,B}^{n}\right). \tag{90}$$

This implies that these coefficients are *known* in the iteration $n+1$. Then, it also helps to have a *linear* system.

B.5.2 Algebraic Equation System

I then introduce the definition of forward, backward, and central difference approximations in Eq. (87), and evaluate this equation for every point of the grid. As a result, we have a linear system in the elements of V_j^{n+1}. As a result, Eq. (87) becomes

$$\frac{1}{\Delta}\begin{bmatrix}V_{j=1}^{n+1}\\V_{j=2}^{n+1}\end{bmatrix}-\frac{1}{\Delta}\begin{bmatrix}V_{j=1}^{n}\\V_{j=2}^{n}\end{bmatrix}+\rho\begin{bmatrix}V_{j=1}^{n+1}\\V_{j=2}^{n+1}\end{bmatrix}=\bar{f}^n+\left[\begin{pmatrix}A_1 & 0\\0 & A_2\end{pmatrix}+\begin{pmatrix}-L_1 & L_1\\L_2 & -L_2\end{pmatrix}\right]\begin{bmatrix}V_{j=1}^{n+1}\\V_{j=2}^{n+1}\end{bmatrix} \tag{91}$$

where A_1 and A_2 are coefficient matrices, given by

$$A_1 = \begin{bmatrix}Y_{11} & Z_{11} & 0 & 0 & . & . & 0 & 0\\X_{21} & Y_{21} & Z_{21} & 0 & . & . & 0 & 0\\0 & X_{31} & Y_{31} & Z_{31} & 0 & . & . & 0\\. & & & & & & & \\. & & & & & & & \\. & & & & & & & \\0 & 0 & 0 & . & . & . & X_{I1} & Y_{I1}\end{bmatrix} \tag{92}$$

$$A_2 = \begin{bmatrix}Y_{12} & Z_{12} & 0 & 0 & . & . & 0 & 0\\X_{22} & Y_{22} & Z_{22} & 0 & . & . & 0 & 0\\0 & X_{32} & Y_{32} & Z_{32} & 0 & . & . & 0\\. & & & & & & & \\. & & & & & & & \\. & & & & & & & \\0 & 0 & 0 & . & . & . & X_{I2} & Y_{I2}\end{bmatrix} \tag{93}$$

Furthermore, L_1 and L_2 are defined as

$$L_1 = \begin{bmatrix}\theta\lambda_1 & 0 & 0 & 0 & \cdots & 0 & 0\\0 & \theta\lambda_1 & 0 & 0 & \cdots & \cdots & 0\\0 & 0 & \theta\lambda_1 & 0 & \cdots & \cdots & \cdots\\\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\0 & 0 & 0 & 0 & \cdots & \cdots & \theta\lambda_1\end{bmatrix} \tag{94}$$

$$L_2 = \begin{bmatrix}\theta\lambda_2 & 0 & 0 & 0 & \cdots & 0 & 0\\0 & \theta\lambda_2 & 0 & 0 & \cdots & \cdots & 0\\0 & 0 & \theta\lambda_2 & 0 & \cdots & \cdots & \cdots\\\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\0 & 0 & 0 & 0 & \cdots & \cdots & \theta\lambda_2\end{bmatrix} \tag{95}$$

Then, I express the system (91) in matrix form as follows.

$$\frac{1}{\Delta}(V^{n+1}-V^n)+\rho V^{n+1}=\bar{f}^n+A^nV^{n+1}, \tag{96}$$

where

$$A^n = \left[\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} + \begin{pmatrix} -L_1 & L_1 \\ L_2 & -L_2 \end{pmatrix}\right], \quad \bar{f}^n = \begin{bmatrix} V_{11}^n \, (c_{11}^n)^{1-\delta} \, [(1-\gamma)V_{11}^n]^{-\theta}) \\ \vdots \\ V_{I1}^n \, (c_{I1}^n)^{1-\delta} \, [(1-\gamma)V_{I1}^n]^{-\theta} \\ V_{12}^n \, (c_{12}^n)^{1-\delta} \, [(1-\gamma)V_{12}^n]^{-\theta}) \\ \vdots \\ V_{I2}^n \, (c_{I2}^n)^{1-\delta} \, [(1-\gamma)V_{I2}^n]^{-\theta}) \end{bmatrix}$$

Ordering the terms such as V^{n+1} is on the left side, we have

$$\begin{aligned} \underbrace{\left[\frac{1}{\Delta}\mathrm{I}_{2\mathrm{Ix}2\mathrm{I}} + \rho \mathrm{I}_{2\mathrm{Ix}2\mathrm{I}} - A^n\right]}_{=B^n} V^{n+1} &= \underbrace{\bar{f}^n + \frac{V^n}{\Delta}}_{=b^n} \\ B^n V^{n+1} &= b^n \end{aligned} \tag{97}$$

where B^n is a $(I+1) \times (I+1)$ matrix and b^n is a $(I+1) \times 1$ vector. Both are filled with known coefficients. To find the optimal V^{n+1}, it is common to use a loop in which the difference between V^{n+1} and V^n is less than a convergency criterion. When that criterion is fulfilled, V^{n+1} is the optimal value function.

C Robustness Analysis

In this appendix, we evaluate the model convergency and the existence of equilibrium for a different set of parameter values $(\lambda_1, \lambda_2, y_1, y_2)$.

Table 6: Alternative Parameter Values

Parameters	Symbol	Value
Subjective discount rate	ρ	0.05
Relative risk aversion (RRA)	γ	3
Elasticity of intertemporal substitution (EIS)	$1/\delta$	0.5
Intensity to jump between states	$\{\lambda_1, \lambda_2\}$	$\{0.6931, 0.6931\}$
Borrowing limit	$\underline{a}$	-0.15
Income level	$\{y_1, y_2\}$	$\{0.1, 0.2\}$

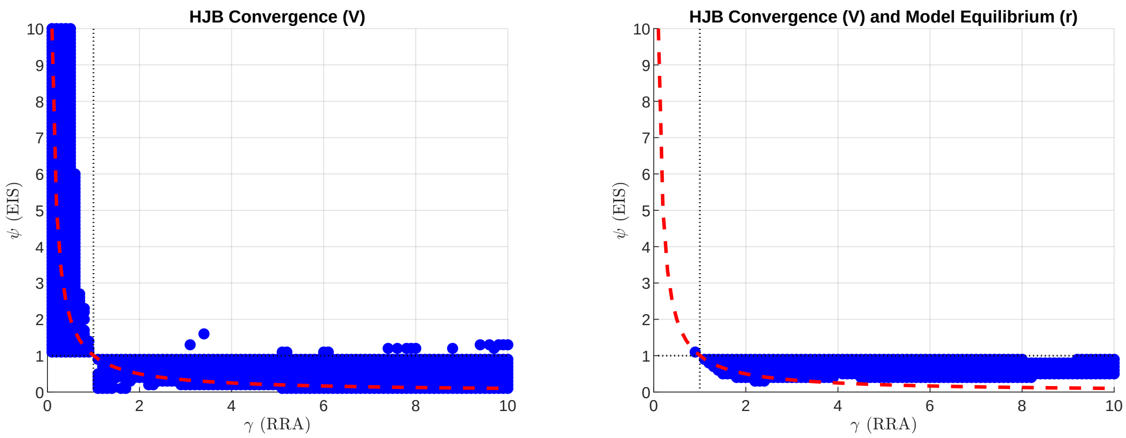

Figure 9: **Model Convergence and Equilibrium.**

D Probabilistic Approach

1. **Type of models to apply this solution technique:** Heterogeneous-agent model with idiosyncratic and aggregate shocks. This model has two characteristics:
   (a) The state variable (a distribution) is *given* when the agents solves his dynamic optimization problem.
      - **State variable:** the distribution of individual characteristics (e.g., wealth distribution)–this is an infinite dimensional state variable (problem: curse of dimensionality).
      - **Challenge:** it is to calculate *conditional expectations* of objects that depends on the state variable (a distribution) when an agent solves his dynamic optimization problem. The solution technique (Probabilistic Solution Technique) attacks this problem: the probabilistic approach enhances the computational efficiency of calculating these expectations.

   (b) The dynamic of the state variable (a distribution) are driven by low-dimensional aggregate shocks ([what is low? two, three aggregate shocks?])

2. **The Probabilistic Solution:** This is a solution technique that searches *global solutions*: It is a solution that accounts for the *full state space* of the model. This contrasts with **local solutions**, which are only valid in the vicinity of a specific state or equilibrium.

3. **Key components of the Probabilistic Solution:**
   - The probabilistic formulation [of What?] indicates that a forward-looking random variable can be expressed as the sum of two components:
     (a) its conditional expectation, and
     (b) a linear impact of exogenous shocks within a short time interval. [Questions: these shocks are the aggregate shocks of the model or what type of shocks they are?]

D.1 An Example

This example illustrates the computational advantage of the probabilistic formulation.

1. **State variable:** (a) one dimensional, and (b) uncontrolled:

$$X_{t+\Delta} = X_t + \mu(X_t)\Delta + \sigma(X_t)(W_{t+\Delta} - W_t), \tag{98}$$

   where Δ denotes the length of time period and $(W_{t+\Delta} - W_t) \sim \mathcal{N}(0, \Delta)$.

2. **The functional equation (e.g. HJB equation):** $V(X_t)$ is the forward-looking process defined as a fixed point of the following functional equation (or the Conditional Expectation Equation):

$$V(X_t) = u(X_t)\Delta + E[V(X_{t+\Delta})|X_t] \tag{99}$$

We want to solve Eq. (99): search $V(X_t)$. We have two approaches.

D.1.1 Approach 1: Analytic Approach

1. Continuous-time: take limit Eq. (98) and (99) when $\Delta \to 0$.
2. Apply Ito's lemma on Eq. (99), and we find the HJB equation: a PDE.
3. We can use finite difference method to approximate the derivatives of V to solve the PDE.

Remark. In the case of one-dimensional state variable, the finite difference method is straightforward to apply. The challenge emerges when X is a distribution (multidimensional state variable).

D.1.2 Approach 2: Probabilistic Approach

1. We rewrite Eq. (99) as follows:

$$E[V(X_{t+\Delta})|X_t] = V(X_t) - u(X_t)\Delta \tag{100}$$

2. Continuous-time: take limit Eq. (98) and (99) when $\Delta \to 0$. Specifically, Eq. (100) becomes:

$$E[V(X_{t+(\Delta\to 0)})|X_t] = V(X_t), \tag{101}$$

   indicating that $V(X_t)$ is a **martingale**.

3. Using the **Martingale Representation Theorem** on $V(X_t)$: when Δ is sufficiently small, there exists a function $z(\cdot)$ such that:

$$V(X_{t+\Delta}) = E[V(X_{t+\Delta})|X_t] + z(X_t)(W_{t+\Delta} - W_t),$$

   which is (the Probabilistic Equation):

$$V(X_{t+\Delta}) = V(X_t) - u(X_t)\Delta + z(X_t)(W_{t+\Delta} - W_t) \tag{102}$$

   - $z(X_t)$: unknown coefficient of the shock.
   - We need to find $z(X_t)$ and $V(X_t)$.
   - Eq. (102) is referred as a **Backward Stochastic Differential Equation (BSDE)**.
   - Eq. (102) holds for any realization of $(W_{t+\Delta} - W_t)$. Then, given X_t, if we have 100 realizations of $(W_{t+\Delta} - W_t)$, we will have 100 equations (102), where $z(X_t)$ and $V(X_t)$ are the variables. So, we have 100 equations and two variables, in this case? Yes. Then, in this case I only need TWO realizations.
   - [Question:] $(W_{t+\Delta} - W_t)$ is the same shock that drives the dynamics of X_t? YES!

- Importantly, $X_{t+\Delta}$ is obtained by Eq. (98):

$$X_{t+\Delta} = X_t + \mu(X_t)\Delta + \sigma(X_t)(W_{t+\Delta} - W_t). \tag{103}$$

- Therefore, we have, in this case, we have two sets of two equations:

$$\begin{aligned}
X_{t+\Delta}^{(1)} &= X_t + \mu(X_t)\Delta + \sigma(X_t)(W_{t+\Delta} - W_t)^{(1)} \\
V(X_{t+\Delta}^{(1)}) &= V(X_t) - u(X_t)\Delta + z(X_t)(W_{t+\Delta} - W_t)^{(1)} \\
X_{t+\Delta}^{(2)} &= X_t + \mu(X_t)\Delta + \sigma(X_t)(W_{t+\Delta} - W_t)^{(2)} \\
V(X_{t+\Delta}^{(2)}) &= V(X_t) - u(X_t)\Delta + z(X_t)(W_{t+\Delta} - W_t)^{(2)}
\end{aligned}$$

However, we are interested in solving the following system:

$$\begin{aligned}
V(X_{t+\Delta}^{(1)}) &= V(X_t) - u(X_t)\Delta + z(X_t)(W_{t+\Delta} - W_t)^{(1)} \\
V(X_{t+\Delta}^{(2)}) &= V(X_t) - u(X_t)\Delta + z(X_t)(W_{t+\Delta} - W_t)^{(2)}
\end{aligned}$$

4. Since these are **Backward** SDE, we can express them as

$$\begin{aligned}
V(X_{t+\Delta}^{(1)}) &= V(X_t) - u(X_t)\Delta + z(X_t)(W_{t+\Delta} - W_t)^{(1)} \\
V(\hat{x}^{(i,j)}) &= V(x^i) - u(x^i)\Delta + z(x^i)(W_{t+\Delta} - W_t)^{(j)}, \quad j = 1, 2, \dots, M. \quad i = 1, 2, 3, \dots, N. \\
V(\hat{x}^{(i,j)}) &= V(x^i) - u(x^i)\Delta + z(x^i)w^{(i,j)}, \qquad (104)
\end{aligned}$$

(105)

- The current state variable X_t can take many values. For instance, we can discretize it in N values as $\{X_t^{(1)}, X_t^{(2)}, \dots, X_t^{(N)}\}$. We denotes the i-th value of X_t as $x^i = X_t^{(i)}$.
- For each current value of the state variable x^i, it is possible to have many realizations of the shock $(W_{t+\Delta} - W_t)^{(j)}$, where $j = 1, 2, \dots, M$. To capture this relationship between x^i and $(W_{t+\Delta} - W_t)^{(j)}$, we denote $(W_{t+\Delta} - W_t)^{(j)}$ as $w^{(i,j)}$. Specifically, $w^{(i,j)}$ denotes the j-th realization of the shock given x^i.
- $\hat{x}^{(i,j)} = X_{t+\Delta}^{(j)}$. It denotes the value of X at $t + \Delta$ given x^i and the shock realization $w^{i,j}$. Then, Eq. (103) can be written as:

$$\hat{x}^{(i,j)} = x^i + \mu(x^i)\Delta + \sigma(x^i)w^{(i,j)}. \tag{106}$$

5. **Approximations:** We approximate $V(\hat{x}^{(i,j)})$ and $z(x^i)$ by "parametric approximation" with parameter represented by Θ as follows:

$$\begin{aligned}
V(\hat{x}^{(i,j)}) &\approx \tilde{V}(\hat{x}^{(i,j)}; \Theta) \qquad (107) \\
z(x^i) &\approx \tilde{z}(x^i; \Theta) \qquad (108)
\end{aligned}$$

Then, in Eq. (104):

$$\tilde{V}(\hat{x}^{(i,j)};\Theta) = \tilde{V}(x^i;\Theta) - u(x^i)\Delta + \tilde{z}(x^i;\Theta)w^{(i,j)} \tag{109}$$

6. **Loss Function.** We then construct a "Loss Function" from Eq. (110):

$$L^{(i,j)}(\Theta) = \tilde{V}(\hat{x}^{(i,j)};\Theta) - \tilde{V}(x^i;\Theta) + u(x^i)\Delta - \tilde{z}(x^i;\Theta)w^{(i,j)} \tag{110}$$

   Ideally, $L^{(i,j)}(\Theta)$ should be zero.

7. **Optimization Problem:** Finding Θ such that we find the approximated solutions: $\tilde{V}(x^i;\Theta)$ and $\tilde{z}(x^i;\Theta)$:

$$\min_{\Theta} \quad \frac{1}{N_x M_w} \sum_{i=1}^{N_x} \sum_{j=1}^{M_w} \left(L^{(i,j)}(\Theta) \right)^2$$

   subject to:

$$\hat{x}^{(i,j)} = x^i + \mu(x^i)\Delta + \sigma(x^i)w^{(i,j)} \tag{111}$$

$$x^i \quad : \quad \text{is the i-th value of } X_t \tag{112}$$

$$w^{(i,j)} \quad : \quad \text{is the j-th shock realization given } x^i \text{ that follows } N(0,\Delta) \tag{113}$$

Two Remarks:

- The volatility term $z(\cdot)$ is crucial in transforming the "conditional expectation equation" to the "probabilistic equation".
- $z(\cdot)$ plays a vital role in solving portfolio choice problems in HA models:
  - $V'(x)\sigma(x) = z(x)$
  - $V'(x)\sigma(x) = z(x)$: it captures the impact of exogenous shocks on agents' lifetime expected utility.

References

Achdou, Yves, Jiequn Han and Jean-Michel Lasry (2022), 'Income and Wealth Distribution in Macroeconomics: A Continuous-Time Approach', *The Review of Economic Studies* p. 42.

Ai, Hengjie, Ravi Bansal, Hongye Guo and Amir Yaron (2023), 'Identifying Preference for Early Resolution from Asset Prices', *NBER Working Paper No. 31087* .

Bansal, Ravi and Amir Yaron (2004), 'Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles', *The Journal of Finance* **59**(4), 1481–1509.

Candler, Graham V. (2001), Finite-Difference Methods for Continuous-Time Dynamic Programming, *in* R.Marimon and A.Scott, eds, 'Computational Methods for the Study of Dynamic Economies', Oxford University Press, pp. 172–194.

Chan, Yeung Lewis and Leonid Kogan (2002), 'Catching Up with the Joneses: Heterogeneous Preferences and the Dynamics of Asset Prices', *journal of political economy* p. 31.

Epstein, Larry G., Emmanuel Farhi and Tomasz Strzalecki (2014), 'How Much Would You Pay to Resolve Long-Run Risk?: The American Economic Review', *The American Economic Review* **104**(9), 2680–2697.

Epstein, Larry G. and Stanley E. Zin (1989), 'Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework', *Econometrica* **57**(4), 937–969.

Fernández-Villaverde, Jesús, Samuel Hurtado and Galo Nuño (2023), 'Financial Frictions and the Wealth Distribution', *Econometrica* **91**(3), 869–901.

Gârleanu, Nicolae and Stavros Panageas (2015), 'Young, Old, Conservative, and Bold: The Implications of Heterogeneity and Finite Lives for Asset Pricing', *journal of political economy* **123**(3), 670–685.

Huggett, Mark (1993), 'The risk-free rate in heterogeneous-agent incomplete-insurance economies', *Journal of Economic Dynamics and Control* **17**(5-6), 953–969.

Kaplan, Greg and Giovanni L. Violante (2018), 'Microeconomic Heterogeneity and Macroeconomic Shocks', *The Journal of Economic Perspectives* **32**(3), 167–194.

Krusell, Per, Toshihiko Mukoyama and Anthony A. Smith (2011), 'Asset prices in a Huggett economy', *Journal of Economic Theory* **146**(3), 812–844.

Longstaff, Francis A. and Jiang Wang (2012), 'Asset Pricing and the Credit Market', *The Review of Financial Studies* **25**(11), 3169–3215.

Pohl, Walter, Karl Schmedders and Ole Wilms (2021), 'Asset pricing with heterogeneous agents and long-run risk', *Journal of Financial Economics* **140**(3), 941–964.

Schneider, Andrés (2022), 'Risk-Sharing and the Term Structure of Interest Rates', *The Journal of Finance* **77**(4), 2331–2374.

Thimme, Julian (2017), 'Intertemporal Substitution in Consumption: A Literature Review', *Journal of Economic Surveys* **31**(1), 226–257.

Wang, Chong, Neng Wang and Jinqiang Yang (2016), 'Optimal consumption and savings with stochastic income and recursive utility', *Journal of Economic Theory* **165**, 292–331.

Wang, Jiang (1996), 'The term structure of interest rates in a pure exchange economy with heterogeneous investors', *Journal of Financial Economics* **41**, 75–110.