A Macro-Finance Model of Capital Reallocation and Misallocation *

Yucheng Zhou†

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Abstract

I study how financial intermediaries' allocation of risk-bearing capacity shapes sectoral investment and capital allocation in a production general-equilibrium model. Leveraged intermediaries allocate wealth across equity claims on sectors that differ in productivity and exposure to aggregate shocks, subject to a constraint on total portfolio risk. I find that sectoral reallocation feeds back into intermediary wealth dynamics and can slow the recovery of output growth after adverse shocks. Binding constraints reinforce this channel by shifting financial risk and real capital away from their allocations in the unconstrained equilibrium. The investment distortions persist over a twenty-year simulation horizon, even though firms face no financing constraints. Binding constraints also redistribute welfare: households incur losses, while intermediaries can gain. The paper highlights intermediaries' role as *risk allocators*, showing how their allocation of risk-bearing capacity across assets shapes sectoral investment and the recovery of output growth.

Keywords: General equilibrium, capital misallocation, financial frictions, feedback effects.
JEL Codes: E13, E32, E44, G11, G12, G21.

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†Gies College of Business, University of Illinois Urbana-Champaign. Email: yz33@illinois.edu.

1 Introduction

The global financial crisis has underscored the importance of the financial system in the real economy. When leveraged intermediaries bear a large share of aggregate risk, adverse shocks to their balance sheets can impair risk bearing and amplify downturns (Brunnermeier and Sannikov, 2014). While we have a good understanding of how financial balance sheets determine the amount of capital investment and economic growth, this raises a question: how does intermediaries' allocation of risk-bearing capacity shape *where* capital is invested across productive sectors?

This question connects to a broader literature on how resource allocation shapes aggregate productivity and economic growth. Misallocation of capital and labor can generate substantial losses in aggregate productivity and output (Restuccia and Rogerson, 2008; Hsieh and Klenow, 2009). Barriers to reallocation can delay the transition to higher productivity (Buera and Shin, 2013) and also slow long-run economic growth (Acemoglu et al., 2018). Where capital is invested also changes risks intermediaries will bear. I ask how this change in capital composition affects the rebuilding of intermediary wealth.

I study these questions in a production general-equilibrium model linking financial risk allocation to sectoral investment. The paper makes two contributions. First, it proposes a risk-sharing channel of capital allocation. In the unconstrained benchmark, sectoral investment and allocation reflect both productivity and compensation for risk endogenously. Binding risk limits create different shadow costs across assets and distort sectoral valuations and investment relative to that benchmark. Second, I show that capital reallocation feeds back into intermediary wealth. When investment shifts toward the safer sector, intermediaries have less exposure to the sectoral risk on which they earn compensation as they rebuild wealth. This feedback operates even in the unconstrained economy, and a binding risk limit can slow recovery further. Together, these findings highlight intermediaries' role as *risk allocators*: their allocation of risk-bearing capacity shapes relative sectoral costs of capital, while the resulting capital composition affects their own wealth rebuilding.

To develop the mechanism formally, I model two productive sectors that differ in productivity and exposure to aggregate risk. Investors differ in risk aversion and face portfolio risk constraints. The multi-sector structure allows financial frictions to affect the sectoral composition of investment as well as its aggregate level. Sectoral capital composition and intermediary capitalization jointly determine valuations, investment, and growth. The model abstracts from firm-level constraints, isolating the effects of constrained financial risk sharing on sectoral investment.

The production side builds on multi-sector models with costly capital adjustment, as in

Eberly and Wang (2025) and Hansen et al. (2024). Capital is sector specific once installed. Changes in relative valuations affect sectoral investment, and adjustment costs make the resulting evolution of capital composition gradual. There is no direct transfer of installed capital between sectors. Both technologies produce the same consumption good. In the quantitative model, sector 1 is more productive and more exposed to aggregate risk than sector 2. Investors hold equity claims on the firms in both sectors. Orthogonal fundamental shocks isolate the comovement in equity returns that arises endogenously through valuations.

Investors have recursive preferences. Financial intermediaries (type A) are relatively risk tolerant, while households (type B) are more risk averse.[1] Both types trade the two equity claims and a risk-free bond. In equilibrium, intermediaries borrow from households and hold leveraged portfolios (He and Krishnamurthy, 2013; Brunnermeier and Sannikov, 2014). Multiple risky assets allow them to choose how to allocate their risk-bearing capacity across sectoral exposures. An institutional counterpart is the allocation of risk limits across trading desks: Bräuning et al. (2026) document spillovers between dealer-bank risk limits in different asset classes.

A VaR-type margin constraint places a bound on the total exposure of each investor to fundamental shocks (Kargar, 2021). Because equity return exposures depend on equilibrium valuations, the restriction on asset holdings varies with the state of the economy. It depends on covariances as well as individual asset volatilities. In the calibration, the constraint occasionally binds for intermediaries while remaining slack for households, consistent with the notion that leverage constraints are most relevant for financial intermediaries. Absent these constraints, investors freely allocate wealth across assets and share aggregate risks, and sectoral investment reflects equilibrium risk-adjusted costs of capital. Misallocation is measured relative to this unconstrained benchmark and arises from restrictions on investors' risk bearing.[2]

Dispersion in marginal products is often used to assess capital misallocation (David and Venkateswaran, 2019). Differences in exposure to aggregate risk can also sustain differences in expected marginal products because investors require compensation for bearing that risk (David et al., 2022). In my model, each sector's marginal product is fixed by its technology. What changes is the compensation investors require to hold its equity. When the intermediary risk limit changes who bears sectoral risk, it changes the price of installed capital in each sector and, through Tobin's Q, firms' investment rates. I measure the resulting investment distortion in each sector as its investment rate under the risk limit minus its rate in the unconstrained

[1] Intermediaries should be interpreted as representing leveraged institutions, such as commercial banks, investment banks, and hedge funds. Households correspond generically to institutions that provide short-term funding.

[2] In the quantitative model, the constraints bind in states with weak intermediary capitalization.

economy at the same capital composition and intermediary wealth share.

The model's equilibrium dynamics are described by two endogenous state variables: x, the share of productive capital in sector 1, and w, intermediaries' share of aggregate wealth. Capital composition determines the mix of fundamental risks on the production side; wealth distribution affects how those risks are shared among investors. Both state variables influence portfolio choices, risk prices, and the valuations that guide investment. Their joint evolution links real capital accumulation to financial wealth dynamics. I solve globally for sectoral capital prices and investors' consumption-wealth ratios, using neural networks in the projection approach of Judd (1992, 2023) to capture how portfolios and investment adjust as the economy moves between regions where the risk limit is slack and where it binds.

Two opposing forces shape capital composition in the unconstrained economy. Consider an adverse shock to sector 1 near the unconstrained steady state, where the sector accounts for most of the capital stock and intermediaries are more exposed to its fundamental shock than households. The shock lowers both x and w. The loss of intermediary wealth raises the price of sector 1's risk and shifts investment toward the safer sector, putting downward pressure on x. The loss of sector 1 capital shifts relative investment back toward sector 1, helping it regain capital share. This change in capital composition also affects how quickly intermediaries rebuild wealth.

With less sector 1 capital, intermediaries face a different mix of risks. They reduce their exposure to the first shock and increase their exposure to the second. They earn less compensation for bearing the dominant sector's risk, and their wealth share tends to recover more slowly. A one-sector model captures the effect of intermediary wealth on risk compensation and aggregate investment. A second sector allows investment decisions to change the composition of the risks on which intermediaries earn that compensation.

When an adverse shock leaves intermediaries sufficiently weakly capitalized, the risk limit binds. In the region where sector 1 dominates, intermediaries bear less of both fundamental risks than in the unconstrained benchmark, with a larger reduction in their exposure to sector 1's shock. To see why, consider adding a dollar of either equity to the intermediary's portfolio. Sector 1 equity loads more heavily on the risk the intermediary already bears, so adding it uses more of the intermediary's scarce risk-bearing capacity. The shadow cost of each position equals the common constraint multiplier times the position's marginal contribution to total risk. Intermediaries therefore tilt their equity holdings toward sector 2.

Households absorb more of sector 1's risk and require greater compensation to bear it. At the distressed state, the prices of both fundamental shocks rise, with a larger increase for sector 1's shock. Relative to the unconstrained equilibrium, the price of installed capital falls in sector 1 and rises in sector 2. Firms respond to these valuations by investing less in sector 1

and more in sector 2, even though they face no financing constraint.

Lower capital growth in sector 1 and higher growth in sector 2 reduce the drift of x relative to the unconstrained economy at the same state. This adds to the downward pressure on sector 1's capital share associated with weak intermediary wealth and works against the investment response that rebuilds the sector after its capital loss. In the low-wealth, high-x region, the constrained economy also has a lower drift of intermediary wealth share. Meanwhile, the investment-induced movement toward lower x leads toward states with weaker expected wealth rebuilding. The interaction of sectoral accumulation and financial wealth thus provides a channel through which investment distortions can affect subsequent financial conditions.

Starting from a state with weak intermediary capitalization and a dominant risky sector, I simulate the constrained and unconstrained economies under the same future fundamental shocks, allowing both x and w to evolve. Intermediaries' mean wealth share recovers more slowly under the risk limit. Sector 1's investment rate remains lower and sector 2's higher on average over the twenty-year horizon. These differences reflect both the constraint's effects on investment and the different paths of capital composition and intermediary wealth. In a separate experiment, I compare how much of the initial decline in aggregate output growth each economy recovers after a one-standard-deviation adverse shock. By year twenty, the unconstrained one-sector economy has recovered about 59 percent of that decline, the unconstrained two-sector economy about 29 percent, and the two-sector economy with the risk limit about 4 percent.

Alternative economies clarify the role of each modeling ingredient. In a one-sector economy with heterogeneous investors, intermediary capitalization and financial constraints affect risk premia and aggregate investment, but there is no capital-composition state and hence no capital reallocation. A multi-sector economy with a representative investor retains sectoral reallocation but leaves little room for constrained risk sharing. The full model combines both state variables, allowing financial constraints to affect both the level and the composition of investment. Asymptotic expansions around simpler economies characterize each margin separately.

When the riskier sector dominates and intermediaries hold very little wealth, both intermediaries and households lose from the calibrated risk limit. At intermediate wealth shares, intermediaries can gain while households continue to lose. Tightening the risk limit widens sectoral investment distortions and deepens household losses. It lowers intermediary welfare when intermediaries hold very little wealth and raises it over an intermediate range of wealth shares. These findings have policy implications for capital requirements. A common risk limit on intermediaries' total risk exposure affects sectors' costs of capital unevenly, and the wel-

fare consequences of tightening the risk limit depend on how wealth is distributed when the requirement is imposed.

The remainder of the paper is organized as follows. Section 2 reviews the literature. Section 3 lays out the model, and Section 4 discusses the model's solution and the characterization of the equilibrium. Section 5 presents the quantitative analysis and discusses the results of the model. Finally, Section 6 concludes.

2 Related literature

This paper fits into several strands of literature. First, it contributes to the voluminous literature on financial frictions and macroeconomic dynamics, beginning with the seminal contributions of Bernanke and Gertler (1989), Bernanke et al. (1999), and Kiyotaki and Moore (1997). I build on the frameworks of He and Krishnamurthy (2013) and Brunnermeier and Sannikov (2014), which link financial-sector conditions to risk premia and asset prices. Recent contributions include Di Tella (2017), Maxted (2024), Krishnamurthy and Li (2025), and Gopalakrishna (2023). My model incorporates heterogeneous investors with recursive preferences and state-dependent portfolio constraints into a multi-sector production economy.

My paper studies how intermediaries allocate scarce risk-bearing capacity across sectoral equity claims and how this allocation shapes firms' investment. Intermediary wealth and the composition of productive capital jointly determine equilibrium portfolios, risk prices, and whether the VaR constraint binds. When the constraint binds, its effect on each equity claim depends on that claim's marginal contribution to portfolio risk. A common portfolio constraint can therefore affect relative valuations and investment across sectors. Productive capital remains sector specific: investors trade financial claims, while physical capital composition adjusts through investment. These investment responses change the risks investors must subsequently absorb, feeding back into equilibrium risk prices and intermediary wealth.

The paper also relates to empirical work showing that intermediary constraints operate across multiple asset markets. He et al. (2017) show that intermediary balance-sheet conditions price a broad cross-section of assets, and recent work provides micro-evidence that dealer constraints generate spillovers across asset markets (Bräuning et al., 2026). Related work on internal capital markets studies the allocation of scarce resources across competing projects (Stein, 1997). Banking studies show how funding and balance-sheet capacity are allocated across units and markets (Cetorelli and Goldberg, 2012a,b; Bräuning and Ivashina, 2020). My model connects intermediaries' allocation of risk-bearing capacity across sectoral equity claims to firms' investment through equilibrium risk prices and valuations.

There is a growing literature studying capital misallocation, early examples of which in-

clude Restuccia and Rogerson (2008) and Hsieh and Klenow (2009). The role of financial frictions on the firm side is studied by Buera et al. (2011), Midrigan and Xu (2014), and Moll (2014), among others. More recently, Ai et al. (2020) develop a contracting theory with intermediation to emphasize the amplifying role of exogenous financial shocks in capital misallocation. Dou et al. (2025) study the GE implications of misallocation through the valuation channel and endogenous growth.

David et al. (2022) consider aggregate risk in a partial equilibrium (PE) setting and show that differences in exposure to aggregate risk can rationalize dispersion in marginal products of capital. van Binsbergen and Opp (2019) study how informational inefficiencies in asset prices distort real capital allocation and highlight the potential gains from correcting mispricing. My paper studies how constrained risk sharing under rational expectations affects capital allocation when risk compensation is determined by investors' portfolio choices and wealth distribution. Investors correctly understand the risks and investment opportunities, but binding portfolio constraints prevent the risk sharing attainable in the unconstrained economy. Improved information or valuation methods alone therefore cannot eliminate the resulting investment distortions. Endogenizing the SDF allows investment to affect equilibrium risk prices by changing the composition of productive capital and the risks investors must absorb. Intermediary wealth and portfolio allocations jointly determine whether the VaR constraint binds and its shadow cost, while each claim's marginal contribution to portfolio risk determines the asset-specific incidence of that cost. The resulting equilibrium valuations guide sectoral investment, feeding back into productive composition and future risk sharing.

On a technical note, this paper also contributes to the heterogeneous-agent asset pricing literature, for which Panageas (2020) provides an excellent survey. The seminal contributions of Dumas (1989, 1992), Wang (1996), Chan and Kogan (2002), Bhamra and Uppal (2009, 2014), Longstaff and Wang (2012), and Gârleanu and Panageas (2015) study equilibrium in economies with two heterogeneous agents and different preference assumptions. Basak and Cuoco (1998), Gârleanu and Pedersen (2011), He and Krishnamurthy (2012, 2013), Chabakauri (2013), and Rytchkov (2014) examine the asset pricing implications of exogenous and endogenous portfolio constraints in economies populated by two heterogeneous agents with one or many assets where constrained agents have logarithmic or constant relative risk aversion (CRRA) preferences. More recent examples include Drechsler et al. (2018), Silva (2020), Kargar (2021), and Schneider (2022), who study the role of heterogeneous agents with recursive preferences in various settings.

This paper extends this line of work to a multi-sector production economy with real frictions in which investors have recursive preferences and face endogenous portfolio constraints. The multi-sector setting draws on the work of Cochrane et al. (2008) and Martin (2013), who

study portfolio rebalancing and return comovement in general equilibrium with multiple Lucas 'trees'. These models assume infinite adjustment costs and do not allow capital movements. At the other extreme, as in Cox et al. (1985), capital adjusts immediately to ensure the balance of portfolios in the absence of adjustment costs. Eberly and Wang (2009, 2025) extend these models by endogenizing the size of 'trees' in a two-sector production economy, allowing for sluggish capital reallocation due to costly adjustment. In related work, Duarte (2018) uses habit preferences in a two-sector production setting to endogenize the risk aversion of a representative agent and shows that this force prolongs economic downturns. Kozak (2022) examines bond-stock comovement in a similar economy in which the representative agent has recursive preferences and exogenously time-varying risk aversion. The multi-sector structure of my model builds on Eberly and Wang (2025) and is related to Duarte (2018) and Kozak (2022) by featuring asymmetric sectors with distinct cash-flow risks. I embed this structure in an economy where heterogeneous investors share risk subject to state-dependent portfolio constraints. Intermediary capitalization affects sectoral investment through the allocation and pricing of risk. Comparing constrained and unconstrained equilibria identifies the investment distortions associated with these constraints and their implications for capital-composition dynamics.

3 Model

In this section, I outline a two-sector general equilibrium model in continuous time. A representative firm in each sector uses capital to produce consumption goods, subject to investment adjustment costs. The real technologies employed by firms are subject to aggregate shocks. The economy is populated by a continuum of investors with recursive preferences, categorized into two classes: financial intermediaries and households. Financial intermediaries are less risk averse than households, although both types of investors have access to the financial market subject to portfolio constraints. The information structure of the economy is standard. Uncertainty is characterized on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$, with a standard filtration $\mathscr{F}_t \subset \mathscr{F}, t \in [0, \infty)$.

In the remainder of this section, I first describe the economic setup and decision problems faced by firms and investors, and then define the competitive equilibrium.

3.1 Production

The productive side of the economy consists of two sectors, each using capital as the sole factor of production. Let K_n, I_n, and Y_n denote the capital stock, gross investment, and output,

respectively, in sector n for $n \in \{1,2\}$. I first introduce the firms' technology, followed by their optimization problems.

3.1.1 Technology

The representative firm in each sector uses an "AK" production technology with constant returns to scale.[3] The stock of capital $K_{n,t}$ produces output at rate

$$Y_{n,t} = A_n K_{n,t} \tag{1}$$

per unit of time, where A_n is a constant governing the productivity of capital.

Capital stock in each sector is exposed to exogenous aggregate Brownian shocks. The capital stock in sector n evolves according to

$$\frac{dK_{n,t}}{K_{n,t}} = \phi_n(\iota_{n,t})dt + \boldsymbol{\sigma}_n' d\boldsymbol{B}_t, \tag{2}$$

where $\iota_{n,t} \equiv I_{n,t}/K_{n,t}$ is the investment-capital ratio in sector n at time t. $\boldsymbol{B} = \{(B_{1,t}, B_{2,t})' \in \mathbb{R}^2; \mathscr{F}_t, t \geq 0\}$ is a two-dimensional Brownian motion on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, representing aggregate shocks to the capital stock. Such shocks may reflect stochastic depreciation, investment-specific shocks, or shocks to capital quality.[4] The vector $\boldsymbol{\sigma}_n$ is the exposure of capital in sector n to the aggregate shocks. In the calibration, I assume no correlation in fundamentals between the two sectors. Thus, capital in sector n is effectively exposed only to $B_{n,t}$. This allows the model to highlight the endogenous correlation between asset returns, rather than the exogenously specified correlation in fundamentals.

Following Hayashi (1982), Jermann (1998), and others, I assume that investment in each sector incurs an adjustment cost that is homogeneous of degree one in investment and the capital stock.[5] The function $\phi(\cdot)$ satisfies $\phi'(\cdot) > 0$ and $\phi''(\cdot) \leq 0$ on the admissible domain. I assume a quadratic adjustment cost function given by

$$\phi(\iota) = \iota - \delta - \frac{\theta}{2}(\iota)^2, \tag{3}$$

where θ is the adjustment cost parameter and δ is the depreciation rate of capital. There is no

[3] See, for example, Kogan (2001, 2004), Eberly and Wang (2025), Pindyck and Wang (2013), and Brunnermeier and Sannikov (2014).

[4] See, for example, Brunnermeier and Sannikov (2014) and Maxted (2024). This paper focuses on a multi-sector setting. The vector notation is used to simplify the exposition.

[5] Homogeneous adjustment cost functions have been widely used in the literature on the q theory of investment. Hayashi (1982) showed that with homogeneous adjustment costs and perfect capital markets, marginal and average q are equal. Jermann (1998) integrated this type of adjustment cost into a real business cycle model.

direct transfer of installed capital between sectors.

3.1.2 Optimization problems

Firms pay output net of investment as dividends to shareholders. The representative firm in each sector chooses the investment rate to maximize the expected discounted value of dividends, given the *stochastic discount factor (SDF)*. The value of the firm in sector n is given by

$$S_{n,t} = \sup_{\iota_{n,s}} \mathbb{E}_t \left[\int_t^\infty \frac{\Lambda_s}{\Lambda_t} (A_n K_{n,s} - \iota_{n,s} K_{n,s}) ds \right]$$
$$\text{s.t. capital evolution (2) and adjustment cost (3).} \tag{4}$$

where Λ_t denotes the SDF that is determined in equilibrium. The evolution of the SDF can be written as

$$\frac{d\Lambda_t}{\Lambda_t} = -r_t dt - \boldsymbol{\eta}_t' d\boldsymbol{B}_t \tag{5}$$

where r_t is the risk-free rate and $\boldsymbol{\eta}_t$ is the vector of market prices of risk, both determined in equilibrium.

3.2 Investors

The economy is populated by a continuum of investors in two classes, A and B. I first describe investors' demographics and preferences, then their budget and financial constraints. Then I define the optimization problems faced by investors.

3.2.1 Demographics and preferences

The aggregate population remains constant and is normalized to one. To obtain stationarity in the model, I follow Gârleanu and Panageas (2015) and consider a simple overlapping-generations (OLG) framework. All investors, regardless of age, preferences, or wealth, face an exogenous and constant mortality rate $\kappa > 0$, and new investors are born at rate κ per unit of time.[6] In aggregate, newborn investors inherit the wealth of their deceased parents on an equal per-capita basis. Of the newly born investors, a constant fraction $\bar{u}$ is of type A, while $1 - \bar{u}$ is of type B. The OLG setup ensures the existence of a stationary distribution of wealth among investors.

[6] The total population size is thus $\int_{-\infty}^{t} \kappa e^{-\kappa(t-s)} ds = 1$.

Investors in both classes have recursive preferences with a constant coefficient of relative risk aversion (RRA) and a constant elasticity of intertemporal substitution (EIS).[7] Specifically, these preferences are described by the stochastic differential utility (SDU) introduced by Duffie and Epstein (1992a,b). That is, an investor of type i for $i \in \{A, B\}$ has lifetime utility defined by

$$J_{i,t} = \mathbb{E}_t \left[\int_t^\infty f_i(C_{i,s}, J_{i,s}) ds \right], \tag{6}$$

where the function $f(C,J)$ is known as the aggregator of the recursive utility for consumption C and the continuation value J. The aggregator function $f(C,J)$ is given by

$$f(C,J) = \frac{1-\gamma}{1-1/\psi} J \left[\left(\frac{C}{[(1-\gamma)J]^{1/(1-\gamma)}} \right)^{1-1/\psi} - (\hat{\rho} + \kappa) \right], \tag{7}$$

where γ denotes RRA and ψ denotes the EIS. For simplicity, assume $\gamma \neq 1$. Time-additive separable CRRA utility is a special case of the above recursive utility specification when the RRA γ is equal to the reciprocal of the EIS ψ, i.e., $\gamma = 1/\psi$. Gârleanu and Panageas (2015) show that, due to mortality risk, the (effective) subjective discount rate is defined as $\rho \equiv \hat{\rho} + \kappa$, where $\hat{\rho}$ denotes the rate of time preference.

The two classes of investors have different coefficients of relative risk aversion γ. I use A to denote financial intermediaries and B to denote households. In particular, households are more risk averse than financial intermediaries, i.e., $\gamma_A < \gamma_B$. It follows that in equilibrium financial intermediaries, including banks, hedge funds, and broker-dealers, raise funds from households via short-term debt financing and use these funds to make levered investments. To focus on heterogeneity in risk aversion, I assume the same EIS across all investors.

3.2.2 Budget and financial constraints

All investors in the economy can trade claims on dividends from the two sectors and a risk-free bond in zero net supply, subject to portfolio constraints. Investors dynamically choose their consumption and portfolio weights. Let s denote the birth date of an investor and $W_{i,t}(s)$ denote the wealth at time t of a type i investor born at time s. Since investors of the same type face the same optimization problems (defined below), their decisions do not depend on the age of each investor. Thus, the notation can be simplified by dropping s in the wealth dynamics. Let $\boldsymbol{\omega}_{i,t}$ be the vector of her portfolio weights in the two risky assets. Then the

[7] See, for example, Kreps and Porteus (1978), Epstein and Zin (1989), and Weil (1989).

financial wealth of an investor of type i evolves according to the dynamic budget constraint

$$\frac{dW_{i,t}}{W_{i,t}} = [\boldsymbol{\omega}_{i,t}'\boldsymbol{\lambda}_{R,t} + r_t - c_{i,t}]dt + \boldsymbol{\omega}_{i,t}'\boldsymbol{\Sigma}_{R,t}d\boldsymbol{B}_t, \tag{8}$$

where $c_{i,t} \equiv C_{i,t}/W_{i,t}$ is the investor's consumption-wealth ratio. r_t is the risk-free rate, $\boldsymbol{\lambda}_{R,t}$ is the vector of expected excess returns on the risky assets, and $\boldsymbol{\Sigma}_{R,t}$ is the matrix of asset return loadings on the aggregate shocks (all defined in equilibrium in Section 4).

The leverage of an investor is defined as the ratio of assets to equity. In equilibrium, intermediaries lever up their balance sheets by borrowing from households. That is, when the sum of portfolio weights in the two risky assets exceeds one, the intermediaries lever up by issuing instantaneous risk-free debt to households. I assume investors face occasionally binding margin constraints, which depend inversely on the volatilities of returns of risky assets. Because return volatilities also depend on the state of the economy, margin requirements are state dependent as well.

I consider a change of variables that transforms the characterization to simplify the exposition. Define the risk exposure of an investor of type i as $\boldsymbol{\sigma}_{i,t}' \equiv \boldsymbol{\omega}_{i,t}'\boldsymbol{\Sigma}_{R,t}$. Given the absence of arbitrage, the vector of expected excess returns of risky assets is given by $\boldsymbol{\lambda}_{R,t} = \boldsymbol{\Sigma}_{R,t}\boldsymbol{\eta}_t$. The dynamic budget constraint can be rewritten as

$$\frac{dW_{i,t}}{W_{i,t}} = [\boldsymbol{\sigma}_{i,t}'\boldsymbol{\eta}_t + r_t - c_{i,t}]dt + \boldsymbol{\sigma}_{i,t}'d\boldsymbol{B}_t. \tag{9}$$

Following the change of variables, I impose the following financial constraint on risk exposure

$$\|\boldsymbol{\sigma}_{i,t}\| \leq \zeta, \tag{10}$$

where $\|\cdot\|$ denotes the norm operator and measures the total risk exposure of investors. ζ denotes the tightness of the constraint on the total risk exposure.[8] Therefore, the investors effectively face a VaR rule, as in Danielsson et al. (2012), Adrian and Shin (2014), and Kargar (2021).[9] I extend the framework to multiple assets and multiple shocks, which allows for a general form of VaR constraints. As will be shown, this friction in the financial market is the key driver of allocative efficiency on the real side of the economy.

[8] This is equivalent to equations (6) and (7) of Kargar (2021), in which the portfolio weight is bounded by $\frac{1}{\alpha\sigma_{R,t}}$ and $\sigma_{R,t}$ denotes the volatility of the risky asset return. The risk exposure is thus bounded by a constant $\frac{1}{\alpha}$. Note that investors will not short the risky asset in equilibrium if there is a single risky asset.

[9] VaR constraints, common for financial institutions in the Basel II and Basel III regulatory frameworks, aim at limiting downside risk and maintaining an equity cushion large enough so that the default probability is kept below some benchmark level.

3.2.3 Optimization problems

Investors within each type face identical optimization problems, as mentioned above. Given initial wealth $W_{i,0} > 0$, they choose their consumption-wealth ratio $c_{i,t}$ and risk exposure $\boldsymbol{\sigma}_{i,t}$ to maximize lifetime utility. The optimization problem of an investor of type i is given by

$$\begin{aligned} V_{i,t} &= \max_{\{c_{i,s}\geq 0, \boldsymbol{\sigma}_{i,s}\}} J_{i,t} \\ &\text{s.t. dynamic budget constraint (9) and financial constraint (10),} \end{aligned} \tag{11}$$

and a solvency constraint $W_{i,t} \geq 0$.

3.3 Equilibrium

The following defines the competitive equilibrium.

Definition 1 *A competitive equilibrium is a set of adapted stochastic processes for the interest rate r_t, market price of risk $\boldsymbol{\eta}_t$, stochastic discount factor Λ_t; firms' output $Y_{n,t}$, capital $K_{n,t}$, investment $I_{n,t}$, and value $S_{n,t}$ for $n \in \{1,2\}$; investors' wealth $W_{i,t}$, consumption $C_{i,t}$, and risk exposure $\boldsymbol{\sigma}_{i,t}$ for $i \in \{A,B\}$ such that*

i. firms solve an optimal investment problem (3.1.2);

ii. investors maximize utility by choosing consumption and risk exposure (3.2.3);

iii. markets clear:

$$\begin{aligned} \bar{u}C_{A,t} + (1-\bar{u})C_{B,t} &= Y_{1,t} + Y_{2,t} - I_{1,t} - I_{2,t} \quad \textit{(goods market)}, \\ \bar{u}W_{A,t} + (1-\bar{u})W_{B,t} &= S_{1,t} + S_{2,t} \quad \textit{(bond market)}, \\ \begin{pmatrix} \boldsymbol{\sigma}_{A,t} & \boldsymbol{\sigma}_{B,t} \end{pmatrix} \begin{pmatrix} \bar{u}W_{A,t} \\ (1-\bar{u})W_{B,t} \end{pmatrix} &= \begin{pmatrix} \boldsymbol{\sigma}_{R_1,t} & \boldsymbol{\sigma}_{R_2,t} \end{pmatrix} \begin{pmatrix} S_{1,t} \\ S_{2,t} \end{pmatrix} \quad \textit{(equity markets)}. \end{aligned} \tag{12}$$

The third condition equates investors' aggregate risk exposures with those supplied by outstanding equity. Using $\boldsymbol{\sigma}_{i,t} = \boldsymbol{\Sigma}_{R,t}'\boldsymbol{\omega}_{i,t}$, it is equivalent to clearing both equity markets when $\boldsymbol{\Sigma}_{R,t}$ is nonsingular.

4 Solution

The equilibrium can be characterized in a recursive formulation, where all equilibrium objects are functions of aggregate state variables. In this section, I first define endogenous state vari-

ables and derive their dynamics. Subsequently, I derive the Hamilton-Jacobi-Bellman (HJB) equations, characterize the solutions to the optimization problems of firms and investors, and provide intuition regarding the optimal policy functions. Then I define a recursive Markov equilibrium and discuss risk-taking decisions and asset pricing in the unconstrained economy. I next explain how equilibrium risk pricing guides capital allocation in both constrained and unconstrained economies. Appendix C uses asymptotic expansions to isolate the roles of sectoral composition and intermediary wealth. The full model with occasionally binding constraints, which is solved numerically, will be discussed in Section 5.

4.1 State variables

Due to the homogeneity of Epstein-Zin preferences and the capital accumulation processes, all variables can be rescaled by total wealth or total capital. Therefore, it suffices to track only two aggregate state variables: the share of aggregate wealth held by type A investors and the share of total capital that belongs to the firm in sector 1. Thus, the equilibrium conditions can be expressed as functions of these two endogenous state variables:

$$w_t = \frac{\bar{u} W_{A,t}}{\bar{u} W_{A,t} + (1-\bar{u}) W_{B,t}}, \qquad x_t = \frac{K_{1,t}}{K_{1,t} + K_{2,t}}. \tag{13}$$

This paper highlights the joint dynamics of w and x as the main drivers of the equilibrium, and the interaction between the two state variables as a key channel through which financial frictions affect the allocation of capital across sectors. The state variable w is one of the key variables in macro-finance models (e.g., He and Krishnamurthy (2013) and Brunnermeier and Sannikov (2014)).[10] In these models, shocks to the aggregate capital stock effectively generate fluctuations in the wealth share of intermediaries, which in turn drive the dynamics of the market price of risk and asset prices (as shown in Appendix D). Thus, the existing macro-finance models do not explicitly model capital reallocation across sectors, which is captured by the state variable x in this model. The variable x has been studied in the literature on production-based models with multiple sectors (e.g., Eberly and Wang (2009), Duarte (2018) and Kozak (2022)). In these models, the representative investor allocates capital across sectors in response to shocks to the sectoral capital stock (as shown in Appendix D).

The following proposition gives the dynamics of the state variable w_t, the wealth share of investors of type A.

[10] Note that the combination of heterogeneity and recursive preferences gives rise to the wealth share as a state variable even under perfect risk sharing.

Proposition 1

$$dw_t = \kappa(\bar{u} - w_t)dt + w_t(1 - w_t)[\mu_{w,t}dt + \boldsymbol{\sigma}'_{w,t}d\boldsymbol{B}_t], \tag{14}$$

where $\mu_{w,t}$ *and* $\boldsymbol{\sigma}_{w,t}$ *are given by*

$$\mu_{w,t} = (\boldsymbol{\sigma}'_{A,t} - \boldsymbol{\sigma}'_{B,t})\boldsymbol{\eta}_t + c_{B,t} - c_{A,t} - (\boldsymbol{\sigma}'_{A,t} - \boldsymbol{\sigma}'_{B,t})\boldsymbol{\Sigma}'_{R,t}\begin{pmatrix} z_t \\ 1 - z_t \end{pmatrix},$$

$$\boldsymbol{\sigma}_{w,t} = \boldsymbol{\sigma}_{A,t} - \boldsymbol{\sigma}_{B,t}.$$

Proof. See Appendix A. ■

Notice that the law of motion of the intermediaries' wealth share w_t has an exogenous component in the drift function due to the demographic structure assumed earlier. This term ensures a non-degenerate long-run distribution of w, i.e., $w \in (0,1)$. The endogenous component of (14) is driven by the consumption and risk exposure choices of investors of both types (in brackets). The term $\mu_{w,t}$ depends on the consumption-wealth ratio of households relative to that of intermediaries, and on the differential aggregate risk exposure between intermediaries and households. The term $\boldsymbol{\sigma}_{w,t}$ only depends on this latter force. As will be shown in (26), the wealth share dynamics depend on asset prices, which themselves depend on wealth share dynamics. The two-way dependence amplifies the return volatility in the two-sector model, extending the result in Brunnermeier and Sannikov (2014).

The two productive sectors serve as a theoretical device to represent a continuum of capital with various degrees of riskiness across all firms in the economy. For exposition, I assume sector 1 is riskier and more productive, while sector 2 is safer but less productive.[11] Thus, sector 1 and sector 2 can be interpreted as the growth sector and the value sector, respectively. Additionally, the fundamental correlation between the capital stocks of the two sectors is assumed to be zero. This is not meant to capture empirical correlations, but it allows me to focus on capital reallocation between two distinct sectors and on the endogenous correlation in asset returns.

Assumption 1 *The risk exposures of sectors 1 and 2 are* $\boldsymbol{\sigma}_1 = (\sigma_1, 0)'$ *and* $\boldsymbol{\sigma}_2 = (0, \sigma_2)'$*, respectively. Sector 1 has greater overall risk exposure because* $\sigma_1 > \sigma_2$*. The productivity of sector 1 is higher than that of sector 2,* $A_1 > A_2$*.*

The following proposition gives the dynamics of the state variable x_t, the share of capital that belongs to the firm in sector 1.

[11] The assumption about productive sectors is a generalization of the assumptions in Duarte (2018) and Kozak (2022).

Proposition 2

$$dx_t = \mu_{x,t}dt + x_t(1-x_t)\sigma_1 dB_{1,t} - x_t(1-x_t)\sigma_2 dB_{2,t}, \tag{15}$$

where the drift of x_t *is given by*

$$\mu_{x,t} = x_t(1-x_t)\left[g_{1,t} - g_{2,t} + (1-x_t)\sigma_2^2 - x_t\sigma_1^2\right].$$

$g_{n,t}$ *denotes the growth rate of capital in sector* n *at time* t.

Proof. See Appendix A. ■

Note that through endogenous capital reallocation, $x_t \in [0,1]$. The endogenous drift term of (15) depends on the relative growth rates of capital in the two sectors, adjusted by the difference in sectoral exposure to shocks. The volatility term is a quadratic function in x which attains its highest value at $x = 1/2$. Meanwhile, $x = 0$ and $x = 1$ correspond to the states where all capital is held by the firm in sectors 2 and 1, respectively.[12] Using the definition of x, the expected growth rate of aggregate capital can be written as

$$g_t = x_t g_{1,t} + (1-x_t)g_{2,t}. \tag{16}$$

It is also useful throughout to define another variable, $z_t \equiv S_{1,t}/(S_{1,t}+S_{2,t})$, the ratio of the firm value in sector 1 to the total value of firms in the two sectors. Variable z_t, not a state variable *per se*, captures the weight of sector 1 in the market portfolio, and is a function of x_t and the relative valuation of firms in the two sectors, as shown later.

4.2 Hamilton-Jacobi-Bellman equations

The recursive formulation of the equilibrium is characterized by the value functions of firms and investors. This section derives the value functions that satisfy the HJB equations.

4.2.1 Firms' problems

The firm's problem in each sector $n \in \{1,2\}$ is to maximize the expected discounted value of dividends $D_{n,t} \equiv A_n K_{n,t} - \iota_{n,t} K_{n,t}$, a problem that can be written recursively as

$$0 = \max_{\iota_{n,t}} \left\{ D_{n,t}dt + \mathbb{E}_t\left[\frac{d(S_{n,t}\Lambda_t)}{\Lambda_t}\right]\right\}. \tag{17}$$

[12] These two states are absorbing states, as the sector cannot be rebuilt once it vanishes, because the loadings on the drift and diffusion terms of the capital stock evolution processes are zero.

The homogeneity property implies that marginal q equals Tobin's average q, following Hayashi (1982). The following proposition gives the firms' value functions.

Proposition 3 *The value of the firm in sector n for $n \in \{1,2\}$ has the form*

$$S_{n,t}(K_{n,t}, x_t, w_t) = K_{n,t} q_{n,t}(x_t, w_t), \tag{18}$$

where $q_{n,t}$ captures future investment opportunities of the firm in sector n. The dynamics of $q_{n,t}$ are given by

$$\frac{dq_{n,t}}{q_{n,t}} = \mu_{q_n,t}\, dt + \boldsymbol{\sigma}'_{q_n,t}\, d\boldsymbol{B}_t, \tag{19}$$

where $(\mu_{q_n,t}, \boldsymbol{\sigma}_{q_n,t})$ is determined in equilibrium. The expressions for $\mu_{q_n,t}$ and $\boldsymbol{\sigma}_{q_n,t}$ are given in Appendix A.

The HJB equation for the firms' problem is given by the following partial differential equation (PDE):

$$0 = \max_{\iota_{n,t}} \left\{ \frac{A_n - \iota_{n,t}}{q_{n,t}} + g_{n,t} + \mu_{q_n,t} + \boldsymbol{\sigma}'_{q_n,t}\boldsymbol{\sigma}_n - r_t - \boldsymbol{\eta}'_t(\boldsymbol{\sigma}_{q_n,t} + \boldsymbol{\sigma}_n) \right\}, \tag{20}$$

where $g_{n,t}$ denotes the expected growth rate of the capital stock in sector n.

Proof. See Appendix A. ■

The firm's optimal investment rate satisfies

$$g'(\iota_{n,t}) = \frac{1}{q_{n,t}}. \tag{21}$$

As in the Q theory of investment, a higher $q_{n,t}$ raises sector n's investment rate.[13]

Sector 1's capital share and its valuation relative to sector 2 determine its weight in the market portfolio,

$$z_t = \left(1 + \frac{q_{2,t}}{q_{1,t}} \frac{1 - x_t}{x_t}\right)^{-1}. \tag{22}$$

The return on holding claims on the firm is given by the dividend yield plus capital gains,

[13] Weighting sectoral valuations and investment rates by each sector's share of physical capital gives aggregate Tobin's Q and the aggregate investment rate,

$$q_t = x_t q_{1,t} + (1 - x_t) q_{2,t}, \qquad \iota_t = x_t \iota_{1,t} + (1 - x_t)\iota_{2,t}.$$

as follows

$$dR_{n,t} = \frac{A_n - \iota_{n,t}}{q_{n,t}} dt + \frac{dS_{n,t}}{S_{n,t}}, \tag{23}$$

$$\equiv \mu_{R_n,t} dt + \boldsymbol{\sigma}'_{R_n,t} d\boldsymbol{B}_t, \tag{24}$$

where

$$\mu_{R_n,t} = \frac{A_n - \iota_{n,t}}{q_{n,t}} + g_{n,t} + \mu_{q_n,t} + \boldsymbol{\sigma}'_{q_n,t} \boldsymbol{\sigma}_n, \tag{25}$$

$$\boldsymbol{\sigma}_{R_n,t} = \boldsymbol{\sigma}_{q_n,t} + \boldsymbol{\sigma}_n. \tag{26}$$

$\mu_{R_n,t}$ and $\boldsymbol{\sigma}_{R_n,t}$ denote the expected return and the vector of volatility terms (loadings on the aggregate shocks) of asset n's return, respectively. Notice that return volatilities are driven by an exogenous component $\boldsymbol{\sigma}_{n,t}$ and an endogenous component $\boldsymbol{\sigma}_{q_n,t}$, which captures the firm's investment opportunities. Define the vector of expected excess returns, $\boldsymbol{\lambda}_{R,t} \equiv \binom{\mu_{R_1,t} - r_t}{\mu_{R_2,t} - r_t}$. Row n of the matrix $\boldsymbol{\Sigma}_{R,t}$ equals $\boldsymbol{\sigma}'_{R_n,t}$. The variance-covariance matrix of the returns is $\boldsymbol{\Omega}_{R,t} = \boldsymbol{\Sigma}_{R,t} \boldsymbol{\Sigma}'_{R,t}$.

4.2.2 Investors' problems

The investor's problem for each type $i \in \{A, B\}$ is to maximize lifetime utility subject to the portfolio risk constraint (10). It can be written recursively as

$$0 = \max_{\substack{c_{i,t} \geq 0 \\ \|\boldsymbol{\sigma}_{i,t}\| \leq \zeta}} \left\{ f_i(c_{i,t} W_{i,t}, V_{i,t}) dt + \mathbb{E}_t[dV_{i,t}] \right\}. \tag{27}$$

With homothetic preferences, the following proposition gives investors' value functions.

Proposition 4 *The value function of investor $i \in \{A, B\}$ has the form*

$$V_{i,t}(W_{i,t}, x_t, w_t) = \frac{W_{i,t}^{1-\gamma_i}}{1-\gamma_i} \xi_{i,t}(x_t, w_t)^{\frac{1-\gamma_i}{1-\psi}}, \tag{28}$$

where $\xi_{i,t}$ captures the type i investor's future investment opportunities. The dynamics of $\xi_{i,t}$ are given by

$$\frac{d\xi_{i,t}}{\xi_{i,t}} = \mu_{\xi_i,t} dt + \boldsymbol{\sigma}'_{\xi_i,t} d\boldsymbol{B}_t, \tag{29}$$

where $(\mu_{\xi_i,t}, \boldsymbol{\sigma}_{\xi_i,t})$ is determined in equilibrium. The expressions for $\mu_{\xi_i,t}$ and $\boldsymbol{\sigma}_{\xi_i,t}$ are given in Appendix A.

The HJB equation for the investors' problem is given by the following partial differential equation (PDE):

$$
\begin{aligned}
0 = \max_{\substack{c_{i,t}\geq 0 \\ \|\boldsymbol{\sigma}_{i,t}\|\leq\zeta}} \Bigg\{ & \frac{1}{1-1/\psi}\left(c_{i,t}^{1-1/\psi}\xi_{i,t}^{1/\psi}-\rho\right)+\left(\boldsymbol{\sigma}_{i,t}'\boldsymbol{\eta}_t+r_t-c_{i,t}\right) \\
& +\frac{1}{1-\psi}\mu_{\xi_i,t}-\frac{\gamma_i}{2}\boldsymbol{\sigma}_{i,t}'\boldsymbol{\sigma}_{i,t} \\
& +\frac{1}{2}\frac{1}{1-\psi}\left(\frac{1-\gamma_i}{1-\psi}-1\right)\boldsymbol{\sigma}_{\xi_i,t}'\boldsymbol{\sigma}_{\xi_i,t} \\
& +\frac{1-\gamma_i}{1-\psi}\boldsymbol{\sigma}_{i,t}'\boldsymbol{\sigma}_{\xi_i,t}\Bigg\}.
\end{aligned} \tag{30}
$$

Proof. See Appendix A. ■

Desired risk exposure is the position investor i would choose if their own risk limit were removed while equilibrium prices and continuation-value dynamics were held fixed. The first-order conditions give the consumption-wealth ratio and desired exposure as

$$c_{i,t} = \xi_{i,t}, \tag{31}$$

$$\boldsymbol{\sigma}_{i,t}^{*} = \frac{1}{\gamma_i}\boldsymbol{\eta}_t + \frac{1}{\gamma_i}\frac{1-\gamma_i}{1-\psi}\boldsymbol{\sigma}_{\xi_i,t}. \tag{32}$$

In equilibrium, the consumption-wealth ratio, which equals $\xi_{i,t}$, responds to changes in investment opportunities. The optimal unconstrained risk exposure has two components, as in the intertemporal capital asset pricing model (ICAPM) result of Merton (1973). The first term in (32) is the myopic demand of a one-period mean-variance investor and is driven by the market price of risk. The intermediaries' lower risk aversion implies a stronger myopic response to changes in the market price of risk. The second term is the hedging demand that captures variation in the investor's investment opportunity set.

Imposing the constraint (10) on the desired exposure (32) yields

$$\boldsymbol{\sigma}_{i,t} = \begin{cases} \boldsymbol{\sigma}_{i,t}^{*}, & \|\boldsymbol{\sigma}_{i,t}^{*}\| \leq \zeta, \\ \zeta\dfrac{\boldsymbol{\sigma}_{i,t}^{*}}{\|\boldsymbol{\sigma}_{i,t}^{*}\|}, & \|\boldsymbol{\sigma}_{i,t}^{*}\| > \zeta. \end{cases} \tag{33}$$

The portfolio objective in (30) is quadratic in risk exposure with curvature $\gamma_i > 0$, so the constrained optimum is the Euclidean projection of desired exposure onto the ball of radius ζ. When desired exposure exceeds ζ, the investor scales all components by the same factor. The following proposition shows how this portfolio-wide constraint affects the marginal cost of

holding each sector's equity.

Proposition 5 *For investor $i \in \{A,B\}$, let $\nu_{i,t} \geq 0$ denote the multiplier on $\zeta - \|\boldsymbol{\sigma}_{i,t}\|$ in the HJB (30). Then*

$$\nu_{i,t} = \gamma_i \max\left\{\|\boldsymbol{\sigma}^*_{i,t}\| - \zeta, 0\right\}, \tag{34}$$

with $\nu_{i,t}(\zeta - \|\boldsymbol{\sigma}_{i,t}\|) = 0$.

For $\|\boldsymbol{\sigma}_{i,t}\| > 0$ and $n \in \{1,2\}$, consider increasing the weight on sector n's equity, holding wealth and the other risky-asset weight fixed and reducing the risk-free bond weight by the same amount. The marginal contribution to investor i's portfolio risk is

$$m_{in,t} \equiv \frac{\partial \|\boldsymbol{\Sigma}'_{R,t}\boldsymbol{\omega}_{i,t}\|}{\partial \omega_{in,t}} = \frac{\boldsymbol{\sigma}'_{R_n,t}\boldsymbol{\sigma}_{i,t}}{\|\boldsymbol{\sigma}_{i,t}\|}. \tag{35}$$

The investor's asset-level first-order condition is

$$\mu_{R_n,t} - r_t = \gamma_i \boldsymbol{\sigma}'_{R_n,t}\boldsymbol{\sigma}_{i,t} - \frac{1-\gamma_i}{1-\psi}\boldsymbol{\sigma}'_{R_n,t}\boldsymbol{\sigma}_{\xi_i,t} + \chi_{in,t}, \tag{36}$$

where $\chi_{in,t} \equiv \nu_{i,t} m_{in,t}$ is the asset-specific constraint term. Consequently,

$$\chi_{i1,t} - \chi_{i2,t} = \nu_{i,t}(m_{i1,t} - m_{i2,t}). \tag{37}$$

At $\boldsymbol{\sigma}_{i,t} = \mathbf{0}$, the constraint is slack, $\nu_{i,t} = 0$, and $\chi_{in,t} = 0$.

Proof. See Appendix A. ■

When investor i's risk limit is slack, $\nu_{i,t} = 0$, so the required premium in (36) reflects only compensation for portfolio risk and an adjustment for hedging value. When the constraint has a positive shadow price, an asset that adds to portfolio risk carries an additional marginal cost, while an asset that reduces portfolio risk frees up capacity. Consider shifting a small portfolio weight from sector 2 to sector 1. If sector 1 contributes more to portfolio risk at the margin, the shift uses scarce capacity at a cost of $\nu_{i,t}(m_{i1,t} - m_{i2,t})$ per unit of weight shifted.

4.3 Markov equilibrium

I define a stationary recursive Markov equilibrium with state variables x_t and w_t, in which all equilibrium objects are expressed as functions of (x, w).

Definition 2 *A Markov equilibrium in state variables (x, w) is a set of functions: equilibrium prices $(r, \boldsymbol{\eta})$, firms' valuation ratio q_n and policy function g_n for $n \in \{1,2\}$, investors' marginal value of wealth ξ_i and policy functions c_i and $\boldsymbol{\sigma}_i$ for $i \in \{A,B\}$ such that*

i. valuation ratio q_n solves the HJB equation for the firm in sector n, and g_n is the corresponding policy function, taking $(r, \boldsymbol{\eta})$ and laws of motion for x and w as given;

ii. marginal value of wealth ξ_i solves the HJB equation for an investor of type i, and c_i and $\boldsymbol{\sigma}_i$ are the corresponding policy functions, taking $(r, \boldsymbol{\eta})$, (q_1, q_2), and laws of motion for x and w as given;

iii. markets clear:

$$wc_A + (1-w)c_B = z\frac{A_1 - \iota_1}{q_1} + (1-z)\frac{A_2 - \iota_2}{q_2} \quad \textit{(goods market)},$$
$$\begin{pmatrix}\boldsymbol{\sigma}_A & \boldsymbol{\sigma}_B\end{pmatrix}\begin{pmatrix} w \\ 1-w\end{pmatrix} = \begin{pmatrix}\boldsymbol{\sigma}_{R_1} & \boldsymbol{\sigma}_{R_2}\end{pmatrix}\begin{pmatrix} z \\ 1-z\end{pmatrix} \quad \textit{(risk exposures market)};$$

iv. the laws of motion for x and w satisfy (15) and (14), respectively.

4.4 Risk taking and asset pricing in the unconstrained economy

While the equilibrium does not permit closed-form solutions even in the absence of financial constraints, this section presents propositions that characterize risk-taking decisions and asset prices in the unconstrained economy.

4.4.1 Investors' risk-taking decisions

The proposition below characterizes intermediaries' exposure to each fundamental shock. Either sector's equity can contribute to both exposures. Intermediaries act as *risk allocators* by choosing the mix of shocks they bear through their sectoral equity positions.

Proposition 6 *In the absence of financial constraints, intermediaries' demand for risk from sector 1, i.e., the first element of their risk exposure $\boldsymbol{\sigma}_A^*$, is given by*

$$\sigma_A^{(1),*} = \frac{zR_w + R_w Q_x x(1-x) - R_x(Q_w + \frac{1}{w})x(1-x) - z\frac{\gamma_B}{w(1-w)} - \frac{\gamma_B}{w(1-w)}x(1-x)Q_x}{(\gamma_B - \gamma_A)Q_w - \frac{\gamma_B}{1-w} - \frac{\gamma_A}{w} + R_w}\sigma_1 \tag{38}$$

where

$$\begin{aligned} R_x &= \frac{1-\gamma_A}{1-\psi}\frac{\xi_{A,x}}{\xi_A} - \frac{1-\gamma_B}{1-\psi}\frac{\xi_{B,x}}{\xi_B}; \quad & R_w &= \frac{1-\gamma_A}{1-\psi}\frac{\xi_{A,w}}{\xi_A} - \frac{1-\gamma_B}{1-\psi}\frac{\xi_{B,w}}{\xi_B}; \\ Q_x &= z\frac{q_{1,x}}{q_1} + (1-z)\frac{q_{2,x}}{q_2}; & Q_w &= z\frac{q_{1,w}}{q_1} + (1-z)\frac{q_{2,w}}{q_2}. \end{aligned} \tag{39}$$

The intermediaries' demand for risk from sector 2, i.e., the second element of their risk exposure $\boldsymbol{\sigma}_A^*$, can be found in Appendix A.

Proof. See Appendix A. ■

Proposition 6 connects investors' risk-sharing incentives to the valuation of productive capital. First, the terms R_x and R_w capture differences in how investors' marginal values of wealth respond to changes in capital composition and wealth distribution, respectively. These differences shape investors' hedging demands and thereby influence how aggregate risks are shared. From (28), the marginal value of wealth is $V_{i,W} \equiv \partial V_i / \partial W_i = W_i^{-\gamma_i} \xi_i^{(1-\gamma_i)/(1-\psi)} > 0$. Holding each investor's own wealth fixed, the risk-sharing terms can therefore be written as

$$R_x = \frac{\partial \log V_{A,W}}{\partial x} - \frac{\partial \log V_{B,W}}{\partial x}, \quad R_w = \frac{\partial \log V_{A,W}}{\partial w} - \frac{\partial \log V_{B,W}}{\partial w}. \tag{40}$$

A positive R_x means that $\log V_{A,W}$ responds more positively to an increase in x than $\log V_{B,W}$; R_w has the analogous interpretation for w. Investors value payoffs especially in states where their marginal value of wealth is high, so differences in these responses create different incentives to hedge changes in investment opportunities. Changing productive composition thus introduces a source of hedging demand through R_x, alongside the wealth distribution channel captured by R_w.

Second, the portfolio revaluation terms Q_x and Q_w capture the value-weighted sensitivities of sectoral valuations to capital composition and wealth distribution, respectively. Under $S_n = K_n q_n$, the price q_n also equals the firm's marginal value of capital. These responses guide firms' investment choices and, together with the diffusion of the state variables, determine the valuation component of aggregate equity risk. Holding each firm's capital stock fixed, the Q terms can therefore be written as

$$Q_x = z \frac{\partial \log S_1}{\partial x} + (1-z) \frac{\partial \log S_2}{\partial x}, \quad Q_w = z \frac{\partial \log S_1}{\partial w} + (1-z) \frac{\partial \log S_2}{\partial w}. \tag{41}$$

The weights z and $1-z$ are the sectors' shares of total equity value. Thus, $Q_w > 0$ means that an increase in intermediary wealth share raises valuations on a value-weighted basis, even if the two sectors respond in opposite directions. The term Q_x captures the corresponding valuation response to capital composition, holding firm capital stocks fixed. Through (26), these valuation responses change the risks carried by the aggregate equity portfolio. Their interaction with investors' hedging incentives links equilibrium risk sharing to the valuation of productive capital, which in turn guides sectoral investment through (21).

It is worth noting that Proposition 6 nests the case of a single productive sector studied in the literature. When there is only one sector (i.e., one Brownian shock), z equals unity, and Q_x and R_x disappear. In this case, there is no capital reallocation, as investors do not have an option to shift risk exposure across sectors, and risk sharing is driven solely by the wealth

distribution channel.

4.4.2 Asset pricing implications

The following propositions present the expressions for the market price of risk $\boldsymbol{\eta}_t$ and the interest rate r_t, which highlight the key implications of asset pricing.

Proposition 7 *In the absence of financial constraints, the market price of risk* $\boldsymbol{\eta}_t$ *can be expressed as follows*

$$\begin{aligned}\boldsymbol{\eta}_t = \gamma_t \left[z_t \boldsymbol{\sigma}_{R_1,t} + (1 - z_t) \boldsymbol{\sigma}_{R_2,t} \right] \\ - \gamma_t \left[w_t \left(\frac{1}{\gamma_A} \right) \left(\frac{1-\gamma_A}{1-\psi} \right) \boldsymbol{\sigma}_{\xi_A,t} + (1 - w_t) \left(\frac{1}{\gamma_B} \right) \left(\frac{1-\gamma_B}{1-\psi} \right) \boldsymbol{\sigma}_{\xi_B,t} \right],\end{aligned} \tag{42}$$

where γ_t *denotes aggregate risk aversion and is given by*

$$\gamma_t = \left(\frac{w_t}{\gamma_A} + \frac{1 - w_t}{\gamma_B} \right)^{-1}. \tag{43}$$

Proof. See Appendix A. ■

The market price of risk in (42) equals aggregate risk aversion times the supply of risk net of investors' hedging demand.[14] Aggregate risk aversion in (43) is the reciprocal of wealth-weighted risk tolerance. When intermediaries hold a smaller share of wealth, aggregate risk aversion is higher and risk prices tend to be higher as well (Panageas, 2020). Capital composition matters for the price of each sectoral shock because it changes the exposure of aggregate wealth to that shock. The second term in (42) captures investors' wealth-weighted hedging demand. When exposure to a particular shock helps investors hedge changes in their investment opportunities, they require less compensation to bear that risk.

Weighting each shock price by the corresponding exposure of each sector's equity return gives the vector of sectoral equity premia,

$$\boldsymbol{\lambda}_{R,t} = \boldsymbol{\Sigma}_{R,t} \boldsymbol{\eta}_t. \tag{44}$$

Although each sector's capital stock is directly exposed to only one fundamental shock, the price of its installed capital can respond to both. The two sectors can therefore have different equity premia even though investors face the same prices of risk. Substituting (42) into (44) gives the two components of each sector's premium. The first reflects the covariance of the

[14] The supply of risk, $z_t \boldsymbol{\sigma}_{R_1,t} + (1 - z_t) \boldsymbol{\sigma}_{R_2,t}$, is the exposure of aggregate wealth to the two shocks.

sector's equity return with the return on aggregate wealth. Equity whose return covaries more strongly with the return on aggregate wealth commands a higher premium. The second captures the equity's hedging value. Investors accept a lower premium on equity that helps them hedge changes in investment opportunities.

Proposition 8 *In the absence of financial constraints, the interest rate r_t can be expressed as follows*

$$\psi\rho + \mu_t + (1-\psi)(r_t + \Phi_t) = z_t \frac{A_1 - \iota_1(g_{1,t})}{q_{1,t}} + (1-z_t)\frac{A_2 - \iota_2(g_{2,t})}{q_{2,t}}, \tag{45}$$

where $\mu_t = w_t\mu_{\xi_A,t} + (1-w_t)\mu_{\xi_B,t}$, and $\Phi_t = w_t\Phi_{A,t} + (1-w_t)\Phi_{B,t}$. The expressions for $\mu_{\xi_i,t}$ and $\Phi_{i,t}$ are given in Appendix A.

Proof. See Appendix A. ■

The left-hand side of (45) represents the (normalized) aggregate demand for goods, while the right-hand side represents the (normalized) aggregate supply of goods. The effects of the interest rate on aggregate demand depend crucially on the EIS: if $\psi > 1$, then an increase in the interest rate decreases aggregate demand (the opposite is true when $\psi < 1$).

4.5 Risk pricing and capital allocation

How does the price of risk shape capital allocation across sectors? A firm invests until the cost of installing one more unit of capital equals its market price. With or without the constraint, a higher $q_{n,t}$ raises sector n's investment rate. Sectoral valuations thus shape the relative growth of the two capital stocks.

Corollary 1 *In either equilibrium, the expected excess return on sector n's equity is $\mu_{R_n,t} - r_t = \boldsymbol{\eta}_t'\boldsymbol{\sigma}_{R_n,t}$. The first expression follows from the HJB of the firm in sector n, evaluated at its optimal investment rate.*

$$\mu_{R_n,t} - r_t = \frac{A_n - \iota_{n,t}}{q_{n,t}} + g_{n,t} + \mu_{q_n,t} + \boldsymbol{\sigma}_{q_n,t}'\boldsymbol{\sigma}_n - r_t. \tag{46}$$

Investor i's portfolio condition in Proposition 5 gives a second expression for the same premium.

$$\mu_{R_n,t} - r_t = \gamma_i\boldsymbol{\sigma}_{R_n,t}'\boldsymbol{\sigma}_{i,t} - \frac{1-\gamma_i}{1-\psi}\boldsymbol{\sigma}_{R_n,t}'\boldsymbol{\sigma}_{\xi_i,t} + \chi_{in,t}. \tag{47}$$

The price of installed capital reconciles the sector's dividend yield and expected capital gains with the return investors require for bearing its risk. Even in the unconstrained equilibrium, sector 1's higher productivity alone cannot establish that it receives too little capital.[15]

[15] This is consistent with the risk-adjusted perspective of David et al. (2022).

When the intermediary's risk limit binds, equity that adds more to its portfolio risk at the margin carries a larger constraint term $\chi_{An,t}$. The constraint can thus change relative valuations and investment rates even though sectoral productivities remain fixed.

Investment in turn changes the risks investors must price. At a given state, faster capital growth in sector 1 relative to sector 2 raises the drift of its capital share, x_t (Proposition 2). As that share grows, aggregate capital becomes more exposed to sector 1 shocks. Meanwhile, portfolio returns and consumption redistribute wealth across investors, changing intermediaries' wealth share w_t and their capacity to bear risk. Future valuations and investment therefore depend on both the mix of risks investors face and who has the wealth to bear them.

4.6 Why multiple sectors and heterogeneous investors?

Two simpler economies illustrate why the model combines multiple sectors, differences in investors' risk tolerance, and a constraint on portfolio risk. In a two-sector economy with a representative investor, the investor bears both shocks. As the capital share of sector 1 changes, so does the mix of risks in the investor's portfolio. Sectoral valuations and investment respond, feeding back into capital composition even though there is no redistribution of risk across investors.

In a one-sector economy with heterogeneous investors, intermediaries bear more of the aggregate shock than households in the unconstrained benchmark. When intermediaries lose wealth, the price of risk rises. A binding portfolio constraint shifts risk toward households and can change aggregate investment. With only one productive sector, all investment adds to the same capital stock.

In the full model, a loss of intermediary wealth can change relative sectoral valuations and investment as a binding risk limit shifts risk toward households. Differences in investment rates then alter the composition of the capital stock and the mix of risks in intermediaries' portfolios. These portfolio exposures complete the feedback by determining how sectoral shocks affect intermediaries' wealth relative to that of households. Appendix C develops the analytical comparisons. Appendix D presents global solutions for the one-sector models with heterogeneous investors and the two-sector model with a representative investor.

5 Quantitative analysis

This section examines how intermediary wealth and capital composition jointly shape sectoral investment. I solve the model's system of PDEs numerically using a global solution method based on neural networks. After describing the numerical method and calibration, I compare

the constrained economy with the unconstrained benchmark and trace how investment distortions change capital composition. I then evaluate welfare for intermediaries and households under the calibrated risk limit. Finally, I examine how tightening the constraint changes risk bearing, sectoral investment, and welfare.

5.1 Numerical solution and calibration

I characterize the equilibrium analytically as a coupled system of second-order PDEs in two state variables, sector 1's share of capital x and intermediaries' share of wealth w. The unknown functions are the prices of capital $q_n(x,w)$ for $n \in \{1,2\}$ and the consumption-wealth ratios $\xi_i(x,w)$ for $i \in \{A,B\}$. Because the system does not admit a closed-form solution, I solve it numerically. Computing the equilibrium is challenging because the dynamics are nonlinear and the financial constraints occasionally bind. I extend the global projection approach of Judd (1992, 2023) by using feedforward neural networks to approximate q_n and ξ_i over the state space.[16] I express the remaining equilibrium objects in terms of these functions and their derivatives and solve the firms' and investors' HJB equations simultaneously. A global solution is crucial for understanding the joint dynamics of (imperfect) risk sharing and capital allocation, unlike a log-linearized representation around the steady state. In Appendix B, I discuss the algorithm used to solve the system of PDEs.

Table 1 reports the model parameters in annual units. The rate of time preference $\hat{\rho}$ and the mortality rate κ are set to 0.001 and 0.015, respectively, implying an effective discount rate of $\hat{\rho}+\kappa = 0.016$.[17] I set the EIS to $\psi = 1.5$ for all investors, a common value found in the asset pricing and macro-finance literature, to focus on heterogeneity in risk-bearing capacity. In the calibrated economy, the wealth-consumption ratio averages 97 and the annual real risk-free rate averages 2.3%. The RRA of intermediaries (type A) is $\gamma_A = 4$, while households (type B) have $\gamma_B = 21$, preserving higher risk tolerance among intermediaries and investors' preferences for early resolution of uncertainty ($\gamma_i > 1/\psi$). Other values within this range generate similar results. To stabilize intermediaries' share of wealth at moderate levels, I set the population share of type A investors to $\bar{u} = 0.1$. Following Kargar (2021), I set the risk limit on intermediaries' total risk exposure to $\zeta = 0.12$. The value-weighted portfolio of the two sectors' equity has an annual Sharpe ratio of 0.3.

Sector 1 is both more productive and more exposed to fundamental risk, as in Assumption 1. I set sectoral productivities to $A_1 = 0.082$ and $A_2 = 0.066$ and capital volatilities to $\sigma_1 = 0.0475$ and $\sigma_2 = 0.035$. The two sectors share a depreciation rate of $\delta_1 = \delta_2 = 0.0139$.

[16] The networks serve as function approximators, a role often played by Chebyshev polynomials in projection methods.

[17] For instance, the effective discount rate is close to that in Gârleanu and Panageas (2015).

Table 1: Calibration

Parameter	Symbol	Value	Target
Rate of time preference	$\hat{\rho}$	0.001	Literature
Mortality rate	κ	0.015	Literature
Elasticity of intertemporal substitution	ψ	1.5	Literature
RRA of type A investors	γ_A	4	Leverage of intermediaries
RRA of type B investors	γ_B	21	Sharpe ratio
Population share of type A investors	$\bar{u}$	0.1	Literature
Productivity of capital in sector 1	A_1	0.082	Mean output growth
Productivity of capital in sector 2	A_2	0.066	Mean output growth
Volatility of capital in sector 1	σ_1	0.0475	Volatility of output growth
Volatility of capital in sector 2	σ_2	0.035	Volatility of output growth
Depreciation rate of capital in sector 1	δ_1	0.0139	Mean output growth
Depreciation rate of capital in sector 2	δ_2	0.0139	Mean output growth
Adjustment cost of capital in sector 1	θ_1	12.25	Investment-capital ratio
Adjustment cost of capital in sector 2	θ_2	11.5	Investment-capital ratio
Tightness of the margin constraint	ζ	0.12	Literature

This table reports the parameter values used in calibrating the model. The model is calibrated in annual units.

Investment adjustment costs are higher in sector 1, with $\theta_1 = 12.25$ and $\theta_2 = 11.5$. These parameter choices are broadly consistent with the macro-finance literature (Brunnermeier and Sannikov, 2014; Silva, 2020; Hansen et al., 2024). In the calibrated economy, aggregate output growth averages 2.1% per year, with a standard deviation of 3.1%.

5.2 Unconstrained benchmark

How do unconstrained intermediaries allocate risk across sectors? I set $\zeta = \infty$ to establish the benchmark for measuring investment distortions. Let (x^*, w^*) denote the unconstrained steady state, where the drifts of both shares are zero. I first vary intermediaries' wealth share w while holding sector 1's capital share at x^*, then vary x while holding w at w^*.

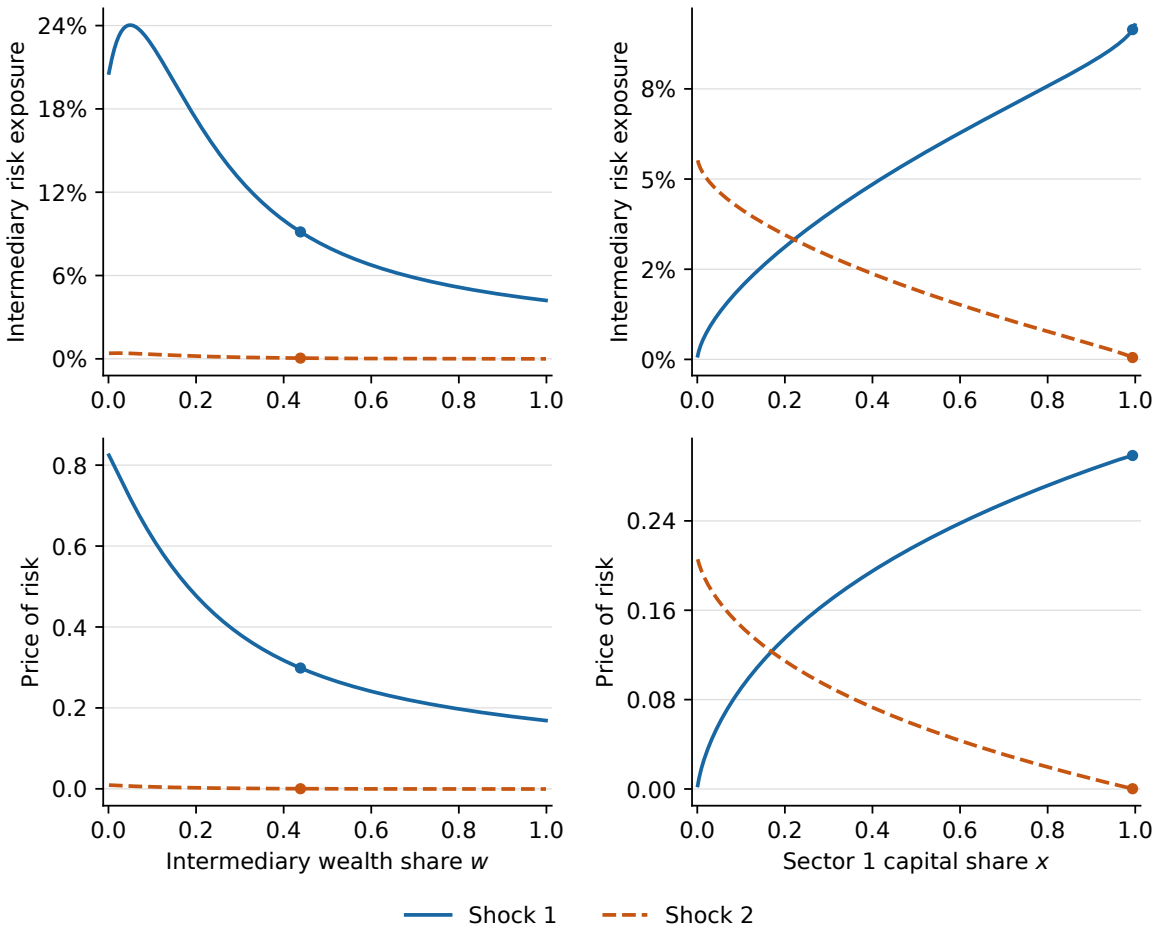
5.2.1 Intermediary risk taking and the pricing of sectoral shocks

Near the steady state, a decline in intermediary wealth shifts risk bearing toward more risk-averse households. The price of sector 1 risk rises, and intermediaries take more exposure to that shock per dollar of their remaining wealth (left column of Figure 1). This is the balance-sheet channel familiar from intermediary asset pricing (He and Krishnamurthy, 2013; Brun-

nermeier and Sannikov, 2014). Intermediaries hold leveraged equity positions and remain more exposed to shock 1 per unit of wealth than households. An adverse sector 1 shock therefore lowers their wealth share on impact, even though sector 2 equity can hedge part of the loss in some states.

Capital composition affects which risks intermediaries bear. Holding $w = w^*$, a decline in sector 1's share reduces the supply of its fundamental risk relative to sector 2's. The price of shock 1 falls and the price of shock 2 rises. Intermediaries reduce their exposure to the first shock and increase their exposure to the second (right column of Figure 1). Each equity can load on both shocks through its valuation, so the figure measures exposure to the shocks directly. Appendix E shows how the underlying equity holdings and return loadings determine the impact on intermediary wealth.

Figure 1: Intermediary exposures and prices of risk in the unconstrained economy



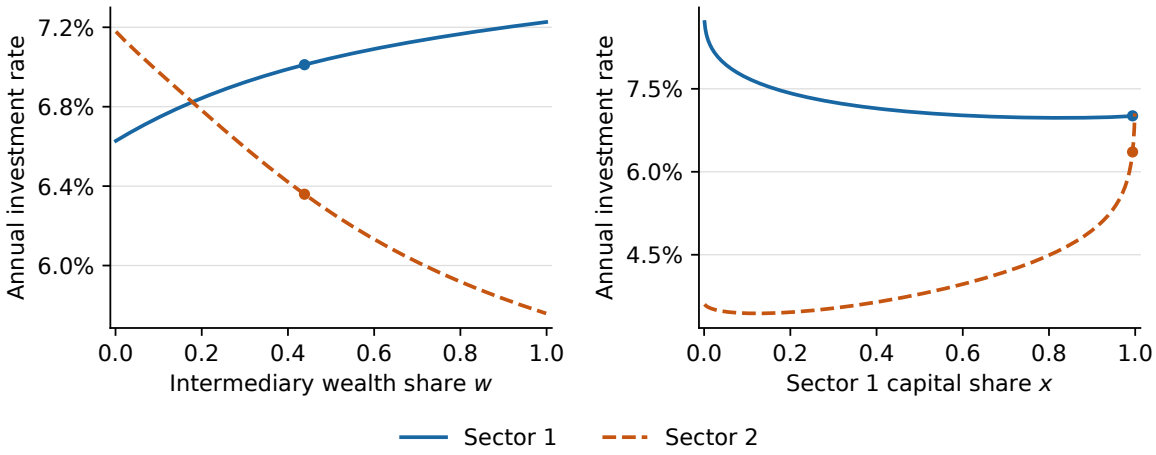
The top panels show intermediaries' signed exposures to the two fundamental shocks per unit of wealth. The bottom panels show the market prices of those shocks. The left panels hold $x = x^*$ and the right panels hold $w = w^*$. Dots mark the unconstrained steady state.

5.2.2 Sectoral investment

How does this allocation of risk affect where firms invest? Sectoral productivities are fixed, but the compensation investors require to hold each sector's equity changes with w and x. These required returns enter the price of installed capital, which determines each firm's investment rate through (21). Thus even the unconstrained allocation of new capital reflects both productivity and risk. When intermediary wealth falls near the steady state and sector 1 dominates the capital stock, its investment rate falls while investment per unit of capital in the safer sector rises (left panel of Figure 2).

A decline in sector 1's capital share shifts relative investment in the other direction. Starting near x^* in the right panel, both investment rates fall as x declines, but the decline is larger in sector 2. Investment therefore shifts toward sector 1 relative to sector 2. Farther from the steady state, when sector 1 accounts for little capital, its claims offer diversification despite its higher fundamental volatility. The price of its risk is then low, and its investment rate exceeds that of sector 2.

Figure 2: Sectoral investment in the unconstrained economy



Each curve reports annual investment per unit of installed capital in the indicated sector. The left panel holds $x = x^*$ and the right panel holds $w = w^*$. Dots mark the unconstrained steady state.

As intermediary wealth falls from its steady-state share, intermediaries seek to take on more risk per dollar of wealth just as firms shift investment toward the safer sector. The next section examines how sectoral investment changes when intermediaries reach their risk limit.

5.3 Capital misallocation

When does the constraint on intermediaries' total risk exposure bind, and how does it change investment across sectors? I compare the economy with the risk limit in (10) to the uncon-

strained equilibrium at the same state (x, w). Both investor types face the constraint, but it binds only for intermediaries in the numerical solution. The top left panel of Figure 3 illustrates the effect when sector 1 accounts for 90 percent of capital. At $w = 0.1$, intermediaries' total risk exposure exceeds 12 percent in the unconstrained economy. With the constraint in place, it reaches $\zeta = 0.12$.

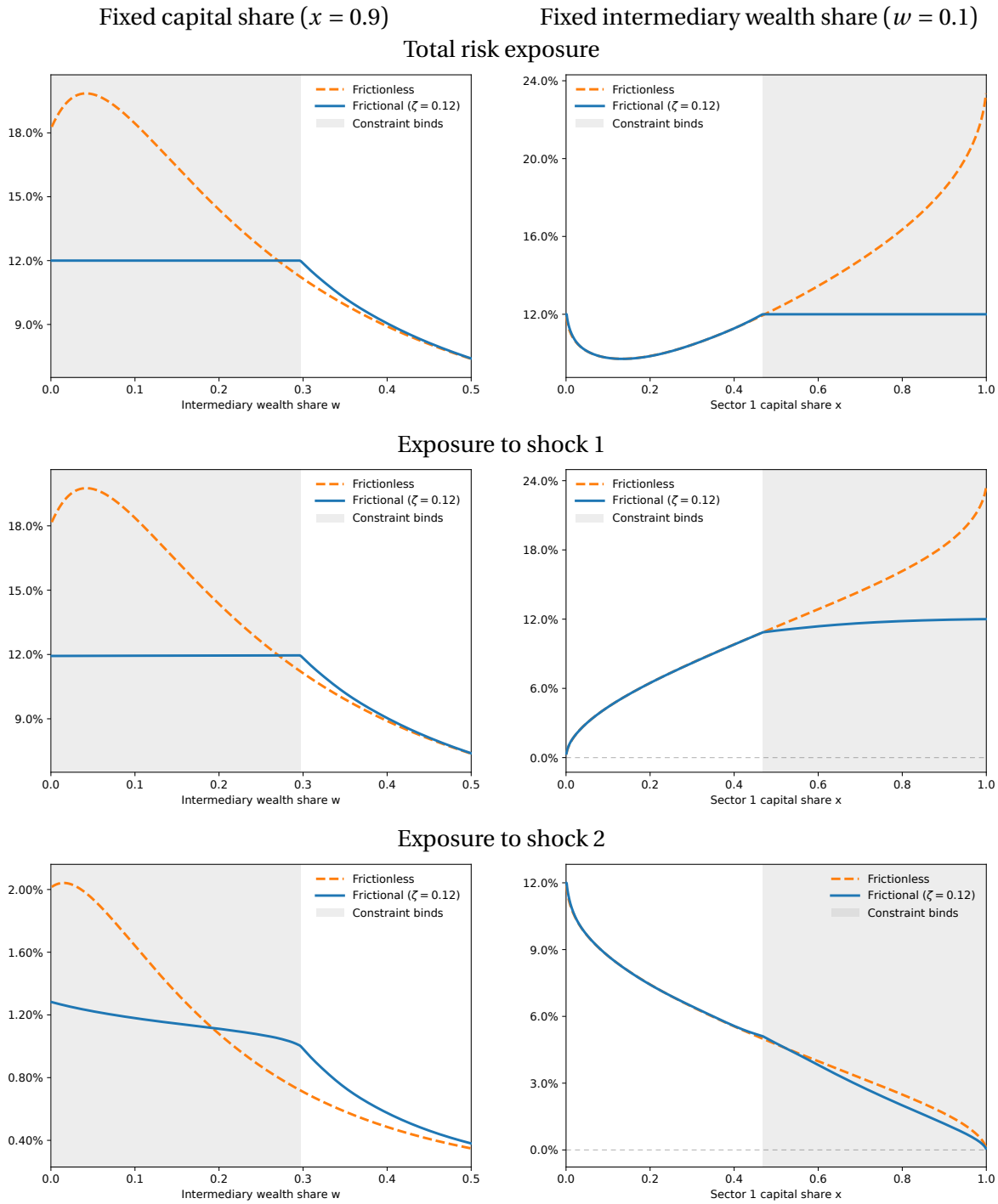
How is that exposure divided between the two fundamental shocks? The second and third rows of Figure 3 show the exposures separately. When sector 1 dominates, intermediaries continue to bear predominantly its risk. At $(x, w) = (0.9, 0.1)$, their exposure to shock 1 is close to 12 percent per unit of wealth, while their exposure to shock 2 is just over 1 percent. Both exposures are lower than in the unconstrained equilibrium, with a larger absolute reduction for shock 1. The right column shows how the mix changes with capital composition at $w = 0.1$. Once the constraint binds, a further increase in sector 1's capital share raises exposure to shock 1 and lowers exposure to shock 2 while their total risk exposure stays at 12 percent. Each sector's equity loads on both shocks, so a change in a sector's portfolio weight affects both components of intermediary risk.

Why does the restriction on total exposure affect the two equity positions differently? The left panel of Figure 4 reports ν_A, the marginal value to intermediaries of relaxing the risk limit at prevailing prices. Where ν_A is positive, adding to either equity position uses scarce capacity to the extent that the position raises total portfolio risk. Proposition 5 gives the resulting wedge in the intermediary's required premium for equity n as $\chi_{An} = \nu_A m_{An}$. The multiplier is common to both positions. The marginal contribution m_{An} measures how much total exposure changes when the intermediary finances an additional unit of portfolio weight in equity n by reducing its bond position. It depends on how closely the equity's return exposures align with the risks the intermediary already bears.

The right panel plots the difference in these wedges, $\chi_{A1} - \chi_{A2} = \nu_A(m_{A1} - m_{A2})$. It is positive where sector 1 dominates and intermediary wealth is low. In those states, an additional position in sector 1 equity uses more of the intermediary's scarce risk capacity than the same addition to sector 2 equity. This gives intermediaries an incentive to tilt their portfolios toward sector 2. In equilibrium they put less weight on sector 1 equity and more on sector 2 equity than in the unconstrained economy, even as sector 1 remains their principal source of risk.

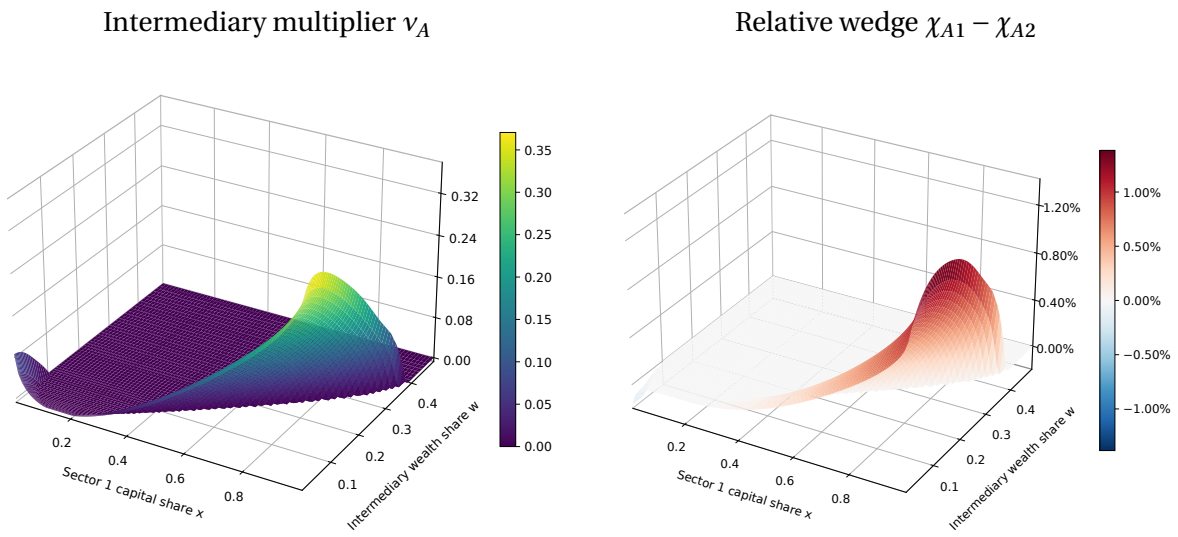
The change in who bears risk matters for its equilibrium price. At $(x, w) = (0.9, 0.1)$, households absorb more of sector 1's fundamental risk as intermediaries reduce their exposure. The more risk-averse households require greater compensation to bear it. The top panels of Figure 5 show that both market prices of risk rise at this state, with a larger increase in η_1. Each sector's equity premium depends on the return's exposures to both shocks and on their prices. The asset-specific wedge χ_{An} describes the intermediary's marginal incentive to hold that eq-

Figure 3: Intermediaries' total risk exposure and its sectoral components



The left column varies intermediary wealth share w and the right column varies sector 1's capital share x. Shock exposures are signed and measured per unit of intermediary wealth. Their Euclidean norm gives intermediaries' total risk exposure. Shading marks states where the risk limit binds.

Figure 4: The shadow value of risk capacity and the relative cost of sector 1 equity

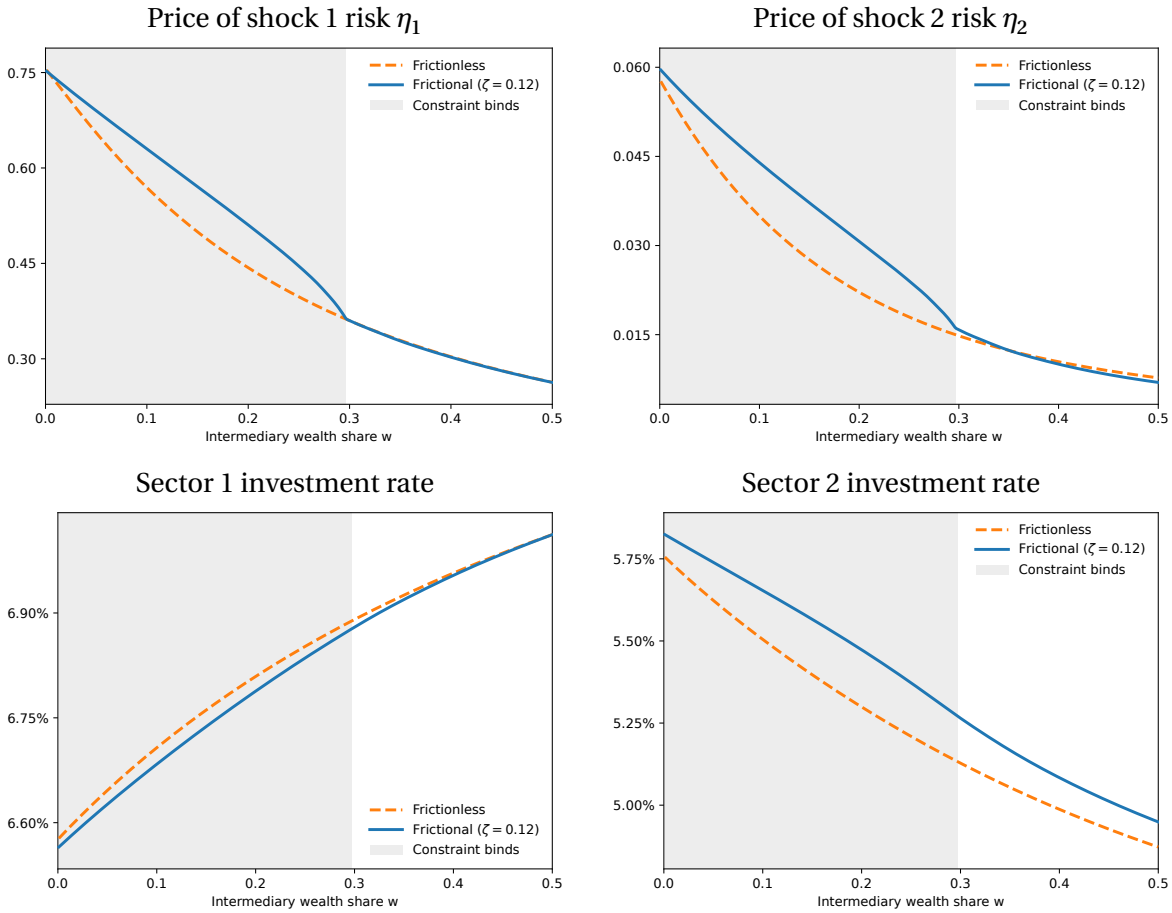


Both panels vary capital composition x and intermediary wealth share w under the constraint $\zeta = 0.12$. The relative wedge measures the difference in the constraint terms in the intermediary's first-order conditions for the two equities.

uity, while the changes in η_1 and η_2 reflect the adjustment in risk sharing across investors.

How do firms respond to the resulting valuations? The bottom panels of Figure 5 report the investment rates directly. I measure the distortion in sector n by $\iota_n^c(x,w) - \iota_n^u(x,w)$, the difference between its constrained and unconstrained investment rates. When sector 1 dominates and intermediaries are weakly capitalized, sector 1's price of installed capital is lower and sector 2's is higher than in the benchmark. Firms respond by investing less in sector 1 and more in sector 2, as implied by (21). The constraint thus changes where new capital is installed through the pricing of sectoral claims, even though firms face no direct financing constraint. This links impaired risk sharing to cyclical capital reallocation (Eisfeldt and Rampini, 2006). Section 5.4 follows how these investment differences feed back into the evolution of capital composition and intermediary wealth.

Figure 5: Sectoral prices of risk and investment under the intermediary risk limit



Each panel varies w at $x = 0.9$ and compares the risk limit of $\zeta = 0.12$ with the unconstrained economy. Shading marks states where the constraint binds. Investment rates are annual investment per unit of installed capital in each sector.

5.4 Feedback between capital allocation and intermediary wealth

How does intermediary wealth shape the allocation of capital, and how does capital allocation in turn shape the recovery of intermediary wealth? The first layer of feedback operates even around the unconstrained steady state. At that state both drifts are zero. When a shock moves the economy away from it, investors adjust their portfolios and firms adjust investment, shaping how both shares evolve. Because intermediaries and households differ in risk tolerance, intermediaries' wealth share w affects the price investors require to bear each sector's risk. The resulting risk prices enter sectoral valuations and hence firms' investment decisions, which move sector 1's capital share x through relative capital accumulation. In turn, x determines the mix of sectoral risks investors must absorb. The exposures intermediaries choose and the compensation they earn on those exposures affect how quickly they rebuild their share of wealth. I use the policy functions from Section 5.2 to trace these two directions of feedback, then examine how a binding risk limit changes both.

Consider an adverse shock to sector 1 starting from the unconstrained steady state. The shock reduces both w and x. Panel A of Table 2 traces each decline separately. With less wealth in intermediary hands, the price of sector 1 risk rises. Intermediaries take larger equity positions per dollar of wealth, and their wealth share tends to recover. This is the balance-sheet recovery force in the unconstrained benchmark. The same change in risk pricing shifts investment toward sector 2, lowering the drift of x. The loss of sector 1 capital has the opposite effect on relative investment. Its price of risk falls, investment shifts toward sector 1 in relative terms, and its capital share tends to recover. Intermediaries then hold less of the dominant sector's equity per dollar of wealth and earn less compensation for its risk, lowering the drift of w.

Table 2: Changes in the drifts of capital and wealth shares

Comparison	Sector 1 capital share Drift of x	Intermediary wealth share Drift of w
Panel A. Unconstrained economy near its steady state		
Lower intermediary wealth share w, fixed x	↓	↑
Lower sector 1 capital share x, fixed w	↑	↓
Panel B. Effect of the risk limit		
Constrained relative to unconstrained	↓	↓

Arrows indicate changes in each share's expected annual change. Panel A varies around the unconstrained steady state. Panel B compares equilibria at $(x, w) = (0.9, 0.1)$, where the risk limit $\zeta = 0.12$ binds.

The second layer of feedback comes from the additional cost of bearing risk in states where the intermediary constraint binds and its multiplier is positive. Panel B of Table 2 shows that the risk limit lowers both drifts at the distressed state. The prices of both shocks rise, and in-

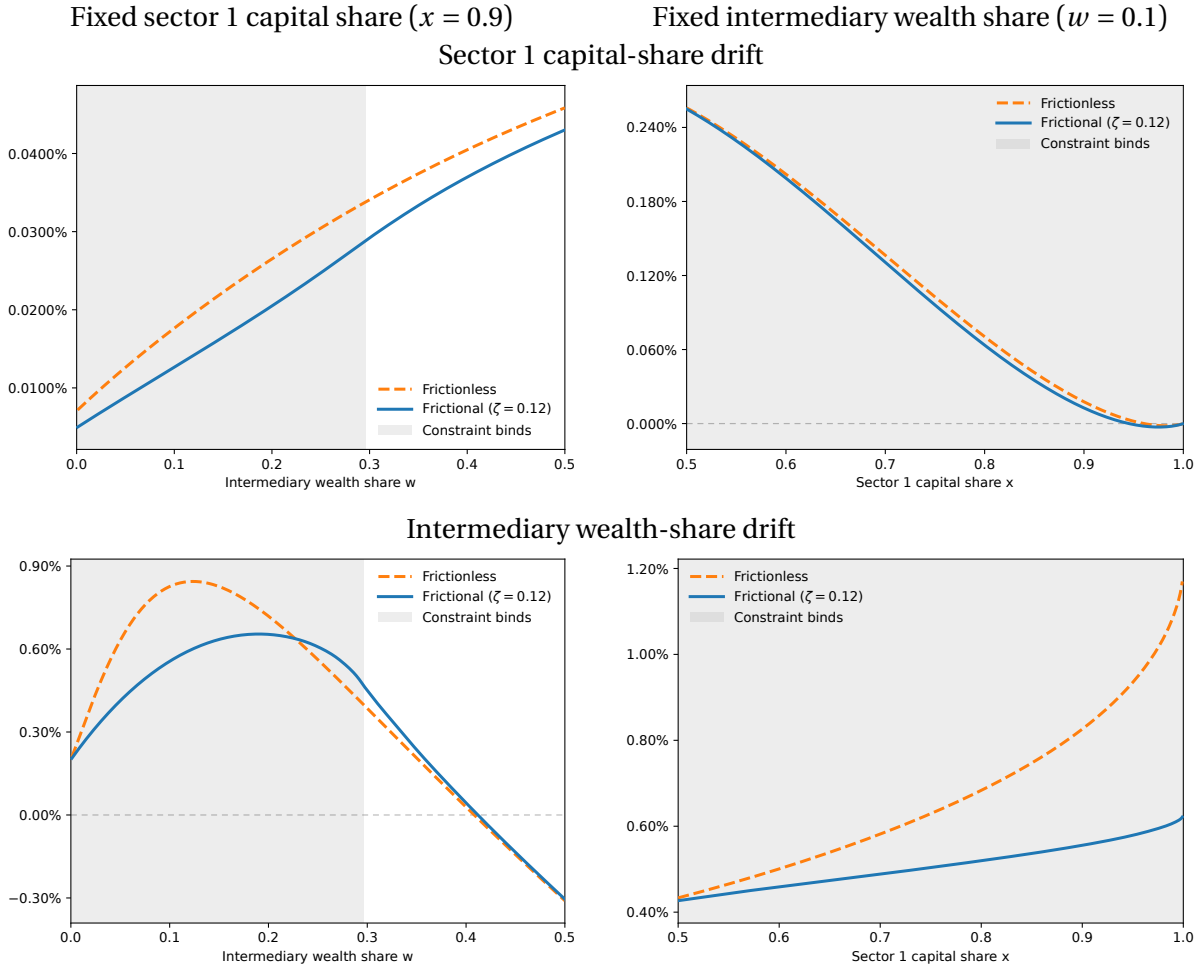
termediaries bear less of each shock per unit of wealth than in the unconstrained economy. Sector 1 investment falls further and sector 2 investment rises, reinforcing the shift toward sector 2 associated with low intermediary wealth. Proposition 2 gives the link to capital accumulation. At a common (x, w), both economies have the same fundamental shock exposures, so the difference in the drift of x is $x(1-x)$ times the difference across economies in relative capital growth, $g_1 - g_2$.

Figure 6 shows how the change in capital allocation feeds back into intermediary wealth. At $x = 0.9$ and $w = 0.1$, both economies have positive capital-share drifts, but sector 1 regains share more slowly under the constraint. The lower right panel shows why this matters for financial recovery. At $w = 0.1$, a smaller share of sector 1 capital is associated with weaker expected wealth rebuilding in both economies. As sector 1's share falls, there is less of the risk on which intermediaries earn compensation. The vertical distance between the curves measures the additional effect of the constraint at the same capital composition. Near $x = 0.9$, the constrained economy has the lower wealth-share drift. The lower left panel also shows how the benchmark recovery force varies with wealth. At $x = 0.9$, a decline in w from intermediate levels raises the unconstrained wealth-share drift until it peaks near $w = 0.12$. Below that point, further wealth losses slow expected rebuilding, although the drift remains positive. At $w = 0.1$, the risk limit lowers this positive drift further.

How long do the investment differences last as both state variables evolve? Figure 7 follows the two economies from the same distressed state under the same sequence of fundamental shocks along each simulated path. Intermediaries regain wealth in both economies, but their mean wealth share recovers more slowly under the risk limit. The difference in investment also persists. After twenty years, sector 1 investment is still about half a percent below its unconstrained benchmark and sector 2 investment is nearly four percent above it. The gaps reflect both the effect of the risk limit on investment at a common state and the different paths of capital composition and intermediary wealth in the two economies.

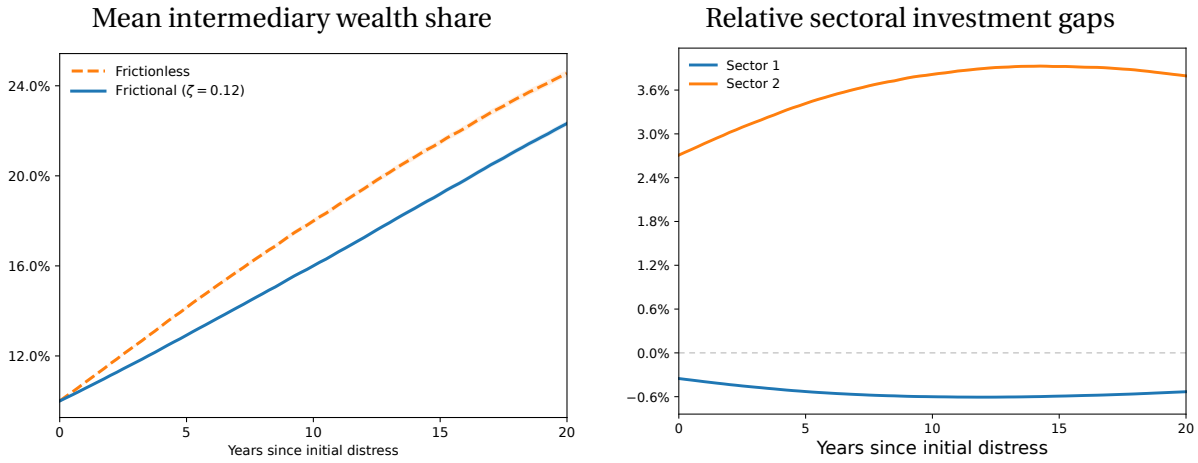
How quickly does aggregate output growth recover after distress? Figure 8 compares the recovery of output growth after a one-standard-deviation adverse shock in three economies. By year twenty, the unconstrained one-sector economy has recovered about 59 percent of the initial decline in growth. In the unconstrained two-sector economy, where investment can shift across sectors, the corresponding recovery is about 29 percent. With the risk limit in place, the two-sector economy has recovered only about 4 percent. In this last economy, growth initially deteriorates further and takes about fifteen years from the start of the shock to regain its first-year rate.

Figure 6: Expected changes in capital composition and intermediary wealth



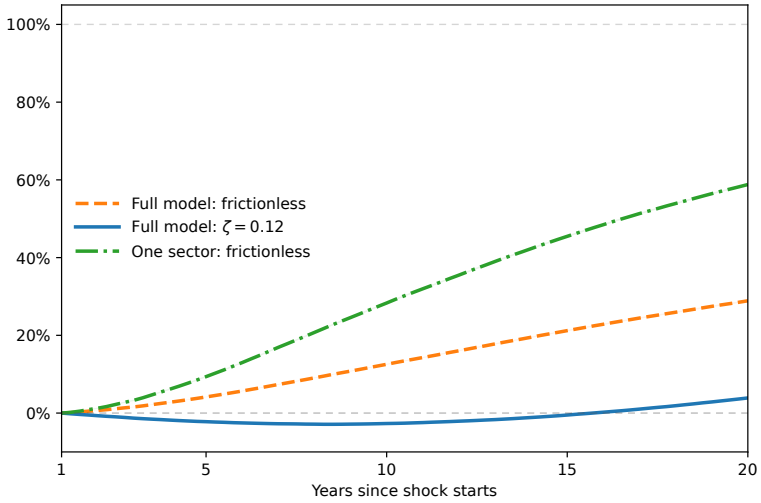
Drifts are expected annual changes in each share, in percentage points. The wealth-share drift includes demographic turnover as in Proposition 1. Shading marks states where the risk limit binds.

Figure 7: Wealth rebuilding and sectoral investment



Years are measured from the common initial state with low intermediary wealth and a dominant sector 1. The right panel reports investment differences in percent relative to the unconstrained economy.

Figure 8: Recovery of aggregate output growth after an adverse shock



The horizontal axis gives years since the shock begins. The vertical axis gives the fraction of the initial decline in output growth recovered in each economy. Zero denotes the first-year response, and 100 percent denotes full recovery of growth. Negative values indicate a further deterioration from the first-year response.

5.5 Welfare effects of the calibrated risk limit

Who gains and who loses when intermediaries face a constraint on portfolio risk? I compare lifetime utility under the risk limit $\zeta = 0.12$ with lifetime utility in the unconstrained economy, starting from the same physical capital stock and the same state (x, w). For investor type i, the consumption equivalent $\Delta_i^{\zeta}(x, w)$ is the proportional change in consumption at every date and in every state of the unconstrained economy that would give the investor the same utility as under the risk limit. A positive value is a welfare gain from the constraint.

Initial wealth of each investor is valued at the capital stock's market value in that economy. At the same initial capital stock and wealth share, a change in the aggregate price of capital $q = xq_1 + (1-x)q_2$ changes each type's initial wealth in the same proportion. Using the homothetic value function in (28), the consumption equivalent satisfies

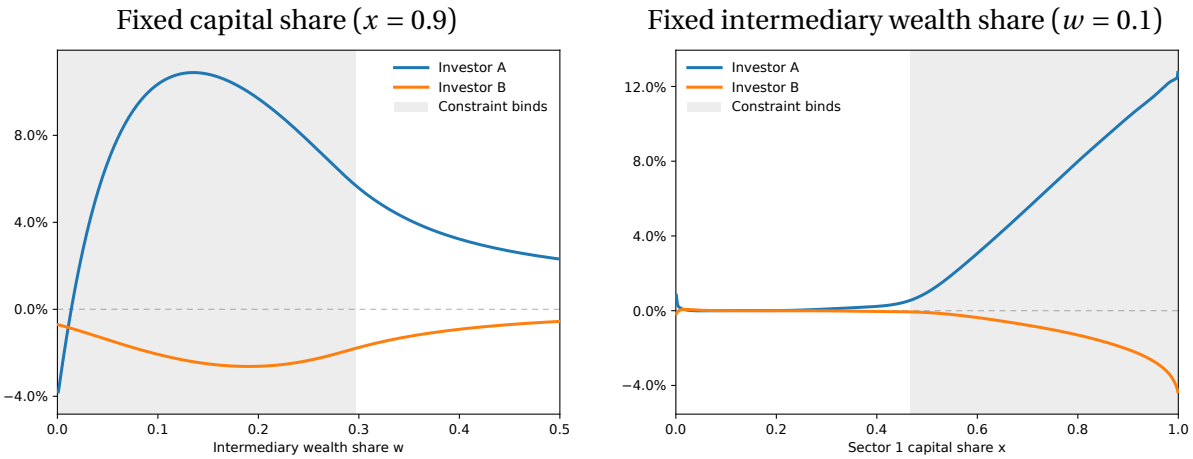
$$1 + \Delta_i^{\zeta}(x, w) = \frac{q^{\zeta}(x, w)}{q^{u}(x, w)} \left(\frac{\xi_i^{\zeta}(x, w)}{\xi_i^{u}(x, w)} \right)^{1/(1-\psi)}, \qquad i \in \{A, B\}, \tag{48}$$

where u and ζ denote the unconstrained and constrained equilibrium, respectively. The first factor captures the revaluation of initial wealth. The second captures the change in the value of each unit of wealth to investor i, including the consumption and investment opportunities the investor faces from that state onward. Appendix A derives the expression.

When sector 1 accounts for a large share of capital ($x = 0.9$) and intermediaries hold very little wealth, both intermediaries and households lose from the risk limit (Figure 9, left panel). At higher initial wealth shares w, intermediaries gain relative to the unconstrained economy, while households continue to lose. The right panel shows how capital composition shapes this welfare redistribution. At $w = 0.1$, the welfare redistribution is small when sector 1 accounts for little capital. As the riskier sector becomes dominant, intermediary gains and household losses widen.

Welfare differences persist beyond the shaded region. In the left panel, intermediaries still gain and households still lose when the constraint is slack, although both effects diminish as intermediary wealth rises. A subsequent loss of intermediary wealth can make the constraint bind, changing who bears sectoral risk and the compensation investors earn. The prospect of those future changes can affect both the price of installed capital today and the value each investor places on a unit of wealth, the two components of the welfare comparison in (48).

Figure 9: Welfare effects of the calibrated risk limit



Curves show consumption equivalents for intermediaries (investor A) and households (investor B), in percent relative to the unconstrained equilibrium. Shading indicates where the intermediary's risk limit binds.

5.6 Effects of tightening the risk limit

How does the tightness of the risk limit affect the allocation of capital and the distribution of welfare? I compare the benchmark $\zeta = 0.12$ with a tighter constraint ($\zeta = 0.10$) and a looser one ($\zeta = 0.15$), evaluating each economy against the same unconstrained benchmark. The top row of Figure 10 shows how much risk intermediaries bear and the shadow cost of the constraint. A tighter constraint binds over a wider range of intermediary wealth shares and raises its shadow cost in the constrained region.

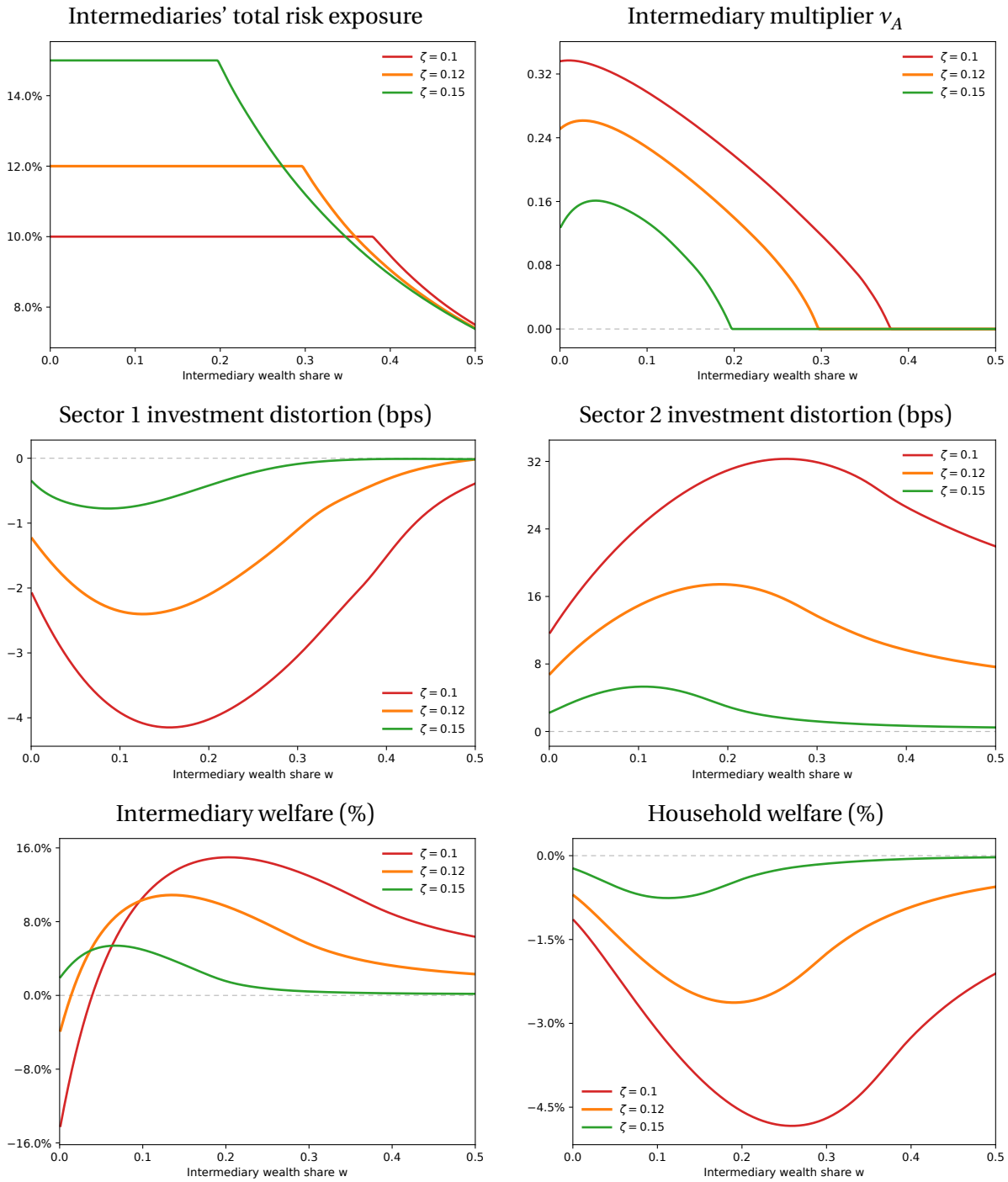
The middle row shows how this change in risk-bearing capacity affects capital allocation. Where sector 1 is dominant and intermediaries are weakly capitalized, tightening the constraint pushes sector 1's investment rate farther below its unconstrained level and sector 2's rate farther above its unconstrained level.

The bottom row shows that tightening the constraint from $\zeta = 0.12$ to 0.10 deepens household welfare losses. For intermediaries, the welfare effect changes sign with their initial wealth share. The tighter constraint lowers their welfare when they hold very little wealth and raises it over an intermediate range of wealth shares.

6 Conclusion

This paper shows how intermediaries, through their role as risk allocators, shape where productive capital is invested. Their portfolio choices determine how sectoral risks are shared with households. The compensation investors require to bear those risks enters the prices

Figure 10: Effects of tightening the intermediary risk limit



Each panel varies intermediary wealth share w at $x = 0.9$ for risk limits $\zeta = 0.10$, 0.12, and 0.15. Investment distortions are differences in annual investment rates relative to the unconstrained economy, in basis points. Welfare is the consumption equivalent relative to that economy, in percent.

of installed capital and guides firms' investment across sectors. When the riskier, more productive sector accounts for most of the capital stock, an additional position in its equity contributes more to intermediary portfolio risk than an equal position in the safer sector. A binding constraint on total exposure then shifts more of the dominant sector's risk to households and raises the price of bearing it. Relative to the unconstrained equilibrium at the same state, firms invest less in the riskier sector and more in the safer sector.

Where capital is invested today also shapes the risks on which intermediaries earn compensation tomorrow. Even in the unconstrained economy, an adverse shock to the dominant sector sets two forces in motion. The loss of sectoral capital encourages firms to rebuild it, while the loss of intermediary wealth shifts investment toward the safer sector. A binding risk limit reinforces the shift toward the safer sector. The riskier sector regains its capital share more slowly, leaving less of its risk for intermediaries to bear as they rebuild wealth. In simulations that start from the same distressed state and face the same fundamental shocks, intermediaries' mean wealth share recovers more slowly under the constraint. After twenty years, the mean investment rate is still about half a percent lower in the riskier sector and nearly four percent higher in the safer sector than in the unconstrained economy.

These findings have policy implications for capital requirements that constrain intermediaries' total risk exposure. A common requirement affects sectors' costs of capital unevenly because their equity claims make different demands on intermediaries' risk-bearing capacity. Its welfare consequences also depend on who holds wealth. When the riskier sector dominates and intermediaries hold very little wealth, both intermediaries and households lose from the calibrated constraint. When intermediaries hold a moderate share of wealth, their interests diverge. Tightening the constraint can widen sectoral investment gaps and deepen household welfare losses even as it raises the lifetime welfare of the intermediaries whose portfolios it restricts.

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Appendices

A Proofs

This appendix proves the propositions in the main text and the lemmas in Appendix C. To simplify notation, time subscripts are dropped when convenient, though it should be understood that they apply.

A.1 Proposition 1

Proof. Let $W_{i,t}(s)$ denote the time-t wealth of an individual investor of type i born at time s. The law of motion of $W_{i,t}(s)$ is given by the budget constraint (9)

$$\frac{dW_{i,t}(s)}{W_{i,t}(s)} = \mu_{i,t}(s)dt + \boldsymbol{\sigma}_{i,t}(s)'d\boldsymbol{B}_t. \tag{49}$$

As investors of the same type face the same optimization problem, regardless of their birth cohort, we have $\mu_{i,t}(s) = \mu_{i,t}$ and $\boldsymbol{\sigma}_{i,t}(s) = \boldsymbol{\sigma}_{i,t}$, i.e., the wealth dynamics do not depend on s (when they were born).

Define the time-t average wealth of all investors of type i, across birth cohorts s, as $W_{i,t} = \int_{-\infty}^{t} \kappa e^{-\kappa(t-s)} W_{i,t}(s)ds$. Applying Itô's lemma gives the following law of motion for $W_{i,t}$

$$dW_{i,t} = \frac{\partial W_{i,t}}{\partial t}dt + \frac{\partial W_{i,t}}{\partial W_{i,t}(s)}dW_{i,t}(s) \tag{50}$$

Applying the Leibniz rule to the first term of (50), we have

$$\begin{aligned}\frac{\partial W_{i,t}}{\partial t} &= \kappa W_{i,t}(t) - \kappa \int_{-\infty}^{t} \kappa e^{-\kappa(t-s)} W_{i,t}(s)ds \\ &= \kappa W_{i,t}(t) - \kappa W_{i,t},\end{aligned} \tag{51}$$

using the definition of $W_{i,t}$. The last term of (50) can be written as

$$\begin{aligned}\frac{\partial W_{i,t}}{\partial W_{i,t}(s)}dW_{i,t}(s) &= \int_{-\infty}^{t} \kappa e^{-\kappa(t-s)} dW_{i,t}(s)ds \\ &= \int_{-\infty}^{t} \kappa e^{-\kappa(t-s)} W_{i,t}(s)ds(\mu_{i,t}dt + \boldsymbol{\sigma}_{i,t}'d\boldsymbol{B}_t) \\ &= W_{i,t}(\mu_{i,t}dt + \boldsymbol{\sigma}_{i,t}'d\boldsymbol{B}_t),\end{aligned} \tag{52}$$

using (49) and the definition of $W_{i,t}$ again.

Substituting these two equations into (50) gives the law of motion for the average wealth

of type i investors

$$dW_{i,t} = \kappa W_{i,t}(t)dt - \kappa W_{i,t}dt + W_{i,t}(\mu_{i,t}dt + \boldsymbol{\sigma}'_{i,t}d\boldsymbol{B}_t). \tag{53}$$

The first term in (53) corresponds to the wealth of newborn agents at time t. Newborns of type i inherit wealth from their parents, who are of type A with probability $\bar{u}$ or type B with probability $1-\bar{u}$. Hence, $W_{i,t}(t) = \bar{u}W_{A,t} + (1-\bar{u})W_{B,t}$.

Therefore, the law of motion of $W_{i,t}$ can be written as

$$dW_{i,t} = \kappa[\bar{u}W_{A,t} + (1-\bar{u})W_{B,t} - W_{i,t}]dt + W_{i,t}(\mu_{i,t}dt + \boldsymbol{\sigma}'_{i,t}d\boldsymbol{B}_t). \tag{54}$$

For $i = A$, the law of motion of $W_{A,t}$ is given by

$$\frac{dW_{A,t}}{W_{A,t}} = \kappa\left(\frac{\bar{u}}{w_t} - 1\right)dt + (\mu_{A,t}dt + \boldsymbol{\sigma}'_{A,t}d\boldsymbol{B}_t). \tag{55}$$

For $i = B$, the law of motion of $W_{B,t}$ is given by

$$\frac{dW_{B,t}}{W_{B,t}} = \kappa\left(\frac{1-\bar{u}}{1-w_t} - 1\right)dt + (\mu_{B,t}dt + \boldsymbol{\sigma}'_{B,t}d\boldsymbol{B}_t). \tag{56}$$

Denote $S_t = S_{1,t} + S_{2,t}$, and use the market clearing condition

$$\bar{u}W_{A,t} + (1-\bar{u})W_{B,t} = S_{1,t} + S_{2,t} = S_t \tag{57}$$

Apply Itô's lemma to the definition of w_t:

$$w_t = \frac{\bar{u}W_{A,t}}{\bar{u}W_{A,t} + (1-\bar{u})W_{B,t}} = \frac{\bar{u}W_{A,t}}{S_t}, \tag{58}$$

we have

$$dw_t = w_t\left[\frac{dW_{A,t}}{W_{A,t}} - \frac{dS_t}{S_t} + \left(\frac{dS_t}{S_t}\right)^2 - \frac{dS_t}{S_t}\frac{dW_{A,t}}{W_{A,t}}\right], \tag{59}$$

in which we can use

$$\frac{dS_t}{S_t} = w_t\frac{dW_{A,t}}{W_{A,t}} + (1-w_t)\frac{dW_{B,t}}{W_{B,t}}. \tag{60}$$

from the definition of S_t. The terms inside the brackets of (59) can be written as

$$\frac{dW_{A,t}}{W_{A,t}} - \frac{dS_t}{S_t} = (1-w_t)\left(\frac{dW_{A,t}}{W_{A,t}} - \frac{dW_{B,t}}{W_{B,t}}\right)$$
$$\left(\frac{dS_t}{S_t}\right)^2 - \frac{dS_t}{S_t}\frac{dW_{A,t}}{W_{A,t}} = -(1-w_t)\left(\frac{dW_{A,t}}{W_{A,t}} - \frac{dW_{B,t}}{W_{B,t}}\right)\left[w_t\frac{dW_{A,t}}{W_{A,t}} + (1-w_t)\frac{dW_{B,t}}{W_{B,t}}\right]$$

Substituting the above two equations into (59) and using (55) and (56), we have

$$\begin{aligned} dw_t = {} & \kappa[\bar{u} - w_t]dt \\ & + (1-w_t)w_t\left[(\boldsymbol{\sigma}'_{A,t} - \boldsymbol{\sigma}'_{B,t})\boldsymbol{\eta}_t + c_{B,t} - c_{A,t}\right]dt \\ & - (1-w_t)w_t\left[(\boldsymbol{\sigma}'_{A,t} - \boldsymbol{\sigma}'_{B,t})\boldsymbol{\Sigma}'_{R,t}\begin{pmatrix} z_t \\ 1-z_t \end{pmatrix}\right]dt \\ & + (1-w_t)w_t(\boldsymbol{\sigma}'_{A,t} - \boldsymbol{\sigma}'_{B,t})d\boldsymbol{B}_t, \end{aligned} \tag{61}$$

The above expression gives the law of motion for w, as shown in the main text. ■

A.2 Proposition 2

Proof. Define $K = K_1 + K_2$, $x = \frac{K_1}{K_1+K_2}$. Use the law of motion of K_1 and K_2

$$\frac{dK_1}{K_1} = g_1 dt + \sigma_1 dB_1,$$
$$\frac{dK_2}{K_2} = g_2 dt + \sigma_2 dB_2.$$

Applying Itô's lemma to K and substituting the definition of K and x, we have

$$\begin{aligned} d(K_1+K_2) &= (g_1K_1 + g_2K_2)dt + \sigma_1 K_1 dB_1 + \sigma_2 K_2 dB_2, \\ dK &= (g_1 xK + g_2(1-x)K)dt + \sigma_1 xK\, dB_1 + \sigma_2 K(1-x)dB_2. \end{aligned} \tag{62}$$

Applying Itô's lemma to x, we have

$$\begin{aligned} dx &= \frac{1}{K}dK_1 - \frac{K_1}{K^2}dK + \frac{K_1}{K^3}(dK)^2 - \frac{1}{K^2}dK_1 dK \\ &= x(g_1 dt + \sigma_1 dB_1) - x[(g_1 x + g_2(1-x))dt + \sigma_1 x dB_1 + \sigma_2(1-x)dB_2] \\ &\quad + x(\sigma_1^2 x^2 + \sigma_2^2(1-x)^2)dt - x\sigma_1^2 x dt \\ &= x(1-x)\left[g_1 - g_2 - \sigma_1^2 x + \sigma_2^2(1-x)\right]dt + \sigma_1 x(1-x)dB_1 - \sigma_2 x(1-x)dB_2. \end{aligned} \tag{63}$$

The above expression gives the law of motion for x, as shown in the main text. ■

A.3 Proposition 3

Proof. To simplify notation, time and sector n subscripts are dropped, as the proposition applies to both sectors at any time.

The HJB equation for the firm's problem is given by

$$0 = \max_{\iota} \left\{ D\,dt + \mathbb{E}\left[\frac{d(\Lambda S)}{\Lambda} \right] \right\}, \tag{64}$$

where D is the dividend paid by the firm, and Λ is the SDF.

We have the following laws of motion

$$\begin{aligned}
\frac{dK}{K} &= g\,dt + \sigma_1 dB_1 + \sigma_2 dB_2, \\
\frac{dq}{q} &= \mu_q dt + \sigma_{q,1} dB_1 + \sigma_{q,2} dB_2, \\
\frac{d\Lambda}{\Lambda} &= -r\,dt - \eta_1 dB_1 - \eta_2 dB_2.
\end{aligned}$$

Applying Itô's lemma to $S = qK$, we have

$$\begin{aligned}
dS &= q\,dK + K\,dq + dq\,dK, \\
\frac{dS}{S} &= \left(g + \mu_q + \sigma_1\sigma_{q,1} + \sigma_2\sigma_{q,2}\right) dt + \left(\sigma_1 + \sigma_{q,1}\right) dB_1 + \left(\sigma_2 + \sigma_{q,2}\right) dB_2, \\
&= \mu_S dt + \sigma_{S,1} dB_1 + \sigma_{S,2} dB_2.
\end{aligned}$$

Applying Itô's lemma to ΛS, we have

$$\frac{d\Lambda S}{\Lambda S} = \left(\mu_S - r - \eta_1\sigma_{S,1} - \eta_2\sigma_{S,2}\right) dt + (\sigma_{S,1} - \eta_1) dB_1 + (\sigma_{S,2} - \eta_2) dB_2. \tag{65}$$

Equating the drift from (65) to D, the HJB equation for the firm's problem is thus given by

$$0 = \max_{\iota} \left\{ \frac{A - \iota}{q} + g + \mu_q + \boldsymbol{\sigma}_q' \boldsymbol{\sigma} - r - \boldsymbol{\eta}'(\boldsymbol{\sigma}_q + \boldsymbol{\sigma}) \right\} \tag{66}$$

where the bold symbols denote vectors for simplicity.

The optimal investment rate is given by the first-order condition

$$g'(\iota) = 1/q. \tag{67}$$

Using Itô's lemma, we can express the drift and diffusion coefficients μ_q, $\sigma_{q,1}$ and $\sigma_{q,2}$ as follows

$$\begin{aligned}\mu_q = &\frac{q_x}{q}\mu_x + \frac{q_w}{q}\left[\kappa(\bar{\mu} - w) + w(1-w)\mu_w\right] \\ &+ \frac{1}{2}\frac{q_{xx}}{q}(\sigma_{x,1}^2 + \sigma_{x,2}^2) + \frac{1}{2}\frac{q_{ww}}{q}w^2(1-w)^2(\sigma_{w,1}^2 + \sigma_{w,2}^2) \\ &+ \frac{q_{xw}}{q}w(1-w)(\sigma_{x,1}\sigma_{w,1} + \sigma_{x,2}\sigma_{w,2}), \qquad (68)\end{aligned}$$

$$\sigma_{q,1} = \frac{q_x}{q}\sigma_{x,1} + \frac{q_w}{q}w(1-w)\sigma_{w,1}, \qquad (69)$$

$$\sigma_{q,2} = \frac{q_x}{q}\sigma_{x,2} + \frac{q_w}{q}w(1-w)\sigma_{w,2}. \qquad (70)$$

where $q_x = \partial q/\partial x$, $q_w = \partial q/\partial w$, $q_{xx} = \partial^2 q/\partial x^2$, $q_{ww} = \partial^2 q/\partial w^2$, $q_{xw} = \partial^2 q/\partial x \partial w$. μ_x, μ_w, $\sigma_{x,1}$, $\sigma_{x,2}$, $\sigma_{w,1}$, $\sigma_{w,2}$ follow from the laws of motion for the state variables x and w, (15) and (14), respectively. ■

A.4 Proposition 4

Proof. To simplify notation, time and investor type i subscripts are dropped, as the proposition applies to investors of both types at any time.

The HJB equation for the investor's problem is given by

$$0 = \max_{\substack{c\geq 0 \\ \|\boldsymbol{\sigma}\|\leq\zeta}} \{f(cW, V)dt + \mathbb{E}[dV]\}, \qquad (71)$$

where f is the aggregator of Epstein-Zin preferences.

We have the following laws of motion

$$\begin{aligned}\frac{dW}{W} &= \mu_W dt + \sigma_{W,1}dB_1 + \sigma_{W,2}dB_2, \\ \frac{d\xi}{\xi} &= \mu_\xi dt + \sigma_{\xi,1}dB_1 + \sigma_{\xi,2}dB_2.\end{aligned}$$

Applying Itô's lemma to

$$V = \frac{W^{1-\gamma}}{1-\gamma}\xi^{\frac{1-\gamma}{1-\psi}},$$

we have

$$dV = \frac{\partial V}{\partial W}dW + \frac{\partial V}{\partial \xi}d\xi + \frac{1}{2}\frac{\partial^2 V}{\partial W^2}(dW)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \xi^2}(d\xi)^2 + \frac{\partial^2 V}{\partial W \partial \xi}dW\,d\xi,$$

in which

$$\begin{aligned}\frac{\mathbb{E}[dV]}{dt} &= (1-\gamma)V\mu_W + \frac{1-\gamma}{1-\psi}V\mu_\xi + \frac{1}{2}(1-\gamma)(-\gamma)V(\sigma_{W,1}^2+\sigma_{W,2}^2)\\ &+ \frac{1}{2}\frac{1-\gamma}{1-\psi}\left(\frac{1-\gamma}{1-\psi}-1\right)V(\sigma_{\xi,1}^2+\sigma_{\xi,2}^2)\\ &+ (1-\gamma)\frac{1-\gamma}{1-\psi}V(\sigma_{W,1}\sigma_{\xi,1}+\sigma_{W,2}\sigma_{\xi,2}).\end{aligned} \tag{72}$$

Meanwhile,

$$f(cW,V) = \frac{1-\gamma}{1-1/\psi}V\left[\left(\frac{cW}{((1-\gamma)V)^{\frac{1}{1-\gamma}}}\right)^{1-1/\psi}-\rho\right] \tag{73}$$

Equating the sum of (72) and (73) to zero and removing the term $(1-\gamma)V$, we have

$$\begin{aligned}0 = \max_{\substack{c\geq 0\\ \|\boldsymbol{\sigma}\|\leq\zeta}}\Bigg\{&\frac{1}{1-1/\psi}\left(c^{1-1/\psi}\xi^{1/\psi}-\rho\right)+[\boldsymbol{\sigma}'\boldsymbol{\eta}_t + r_t - c]\\ &+\frac{1}{1-\psi}\mu_\xi - \frac{\gamma}{2}\boldsymbol{\sigma}'\boldsymbol{\sigma}\\ &+\frac{1}{2}\frac{1}{1-\psi}\left(\frac{1-\gamma}{1-\psi}-1\right)\boldsymbol{\sigma}_\xi'\boldsymbol{\sigma}_\xi\\ &+\frac{1-\gamma}{1-\psi}\boldsymbol{\sigma}'\boldsymbol{\sigma}_\xi\Bigg\},\end{aligned} \tag{74}$$

where matrix notation is used for simplicity. Holding prices and continuation values fixed, the consumption first-order condition and the risk-exposure first-order condition with the portfolio constraint temporarily ignored give

$$c = \xi, \tag{75}$$

$$\boldsymbol{\sigma}^* = \frac{1}{\gamma}\boldsymbol{\eta} + \frac{1}{\gamma}\frac{1-\gamma}{1-\psi}\boldsymbol{\sigma}_\xi. \tag{76}$$

The constrained optimal exposure is given by (33).

Using Itô's lemma, we can express the drift and diffusion coefficients μ_ξ, $\sigma_{\xi,1}$ and $\sigma_{\xi,2}$ as

follows

$$\mu_\xi = \frac{\xi_x}{\xi}\mu_x + \frac{\xi_w}{\xi}\left[\kappa(\bar{\mu} - w) + w(1-w)\mu_w\right] + \frac{1}{2}\frac{\xi_{xx}}{\xi}(\sigma_{x,1}^2 + \sigma_{x,2}^2) + \frac{1}{2}\frac{\xi_{ww}}{\xi}w^2(1-w)^2(\sigma_{w,1}^2 + \sigma_{w,2}^2) + \frac{\xi_{xw}}{\xi}w(1-w)(\sigma_{x,1}\sigma_{w,1} + \sigma_{x,2}\sigma_{w,2}), \tag{77}$$

$$\sigma_{\xi,1} = \frac{\xi_x}{\xi}\sigma_{x,1} + \frac{\xi_w}{\xi}w(1-w)\sigma_{w,1}, \tag{78}$$

$$\sigma_{\xi,2} = \frac{\xi_x}{\xi}\sigma_{x,2} + \frac{\xi_w}{\xi}w(1-w)\sigma_{w,2}. \tag{79}$$

where $\xi_x = \partial\xi/\partial x$, $\xi_w = \partial\xi/\partial w$, $\xi_{xx} = \partial^2\xi/\partial x^2$, $\xi_{ww} = \partial^2\xi/\partial w^2$, $\xi_{xw} = \partial^2\xi/\partial x\partial w$. μ_x, μ_w, $\sigma_{x,1}$, $\sigma_{x,2}$, $\sigma_{w,1}$, $\sigma_{w,2}$ follow from the laws of motion for the state variables x and w, (15) and (14), respectively. ■

A.5 Proposition 5

Proof. Fix an investor type $i \in \{A, B\}$, suppress time subscripts, and define $\mathbf{p}_i \equiv \boldsymbol{\eta} + \frac{1-\gamma_i}{1-\psi}\boldsymbol{\sigma}_{\xi_i}$. Holding prices, continuation values, and aggregate state dynamics fixed, the terms in (30) that depend on the investor's exposure give the Lagrangian

$$\mathscr{L} = \boldsymbol{\sigma}_i'\mathbf{p}_i - \frac{\gamma_i}{2}\|\boldsymbol{\sigma}_i\|^2 + \nu_i\left(\zeta - \|\boldsymbol{\sigma}_i\|\right).$$

The objective is strictly concave and the feasible set is a Euclidean ball with nonempty interior, so the optimality conditions are necessary and sufficient. At a nonzero optimal exposure, differentiating the Lagrangian with respect to $\boldsymbol{\sigma}_i$ gives the first equation below; the remaining conditions impose feasibility and complementary slackness:

$$\mathbf{p}_i - \gamma_i\boldsymbol{\sigma}_i = \nu_i\frac{\boldsymbol{\sigma}_i}{\|\boldsymbol{\sigma}_i\|},$$
$$\|\boldsymbol{\sigma}_i\| \le \zeta, \qquad \nu_i \ge 0, \qquad \nu_i\left(\zeta - \|\boldsymbol{\sigma}_i\|\right) = 0.$$

If the constraint is slack, $\nu_i = 0$ and $\boldsymbol{\sigma}_i = \mathbf{p}_i/\gamma_i$. If $\|\boldsymbol{\sigma}_i\| = \zeta$, the first-order condition for risk exposure implies $\|\mathbf{p}_i\| = \gamma_i\zeta + \nu_i$. Since (32) gives $\mathbf{p}_i = \gamma_i\boldsymbol{\sigma}_i^*$, these cases yield (34). At zero optimal exposure, $\zeta > 0$ implies a slack constraint, so $\nu_i = 0$ and the constraint term vanishes.

Using $\boldsymbol{\sigma}_i = \boldsymbol{\Sigma}_R'\boldsymbol{\omega}_i$, differentiation with respect to ω_{in} gives (35). Premultiplying the first-order condition for risk exposure by $\boldsymbol{\sigma}_{R_n}'$, substituting for $\mathbf{p}_i$, and using $\mu_{R_n} - r = \boldsymbol{\sigma}_{R_n}'\boldsymbol{\eta}$ yields (36) with $\chi_{in} = \nu_i m_{in}$. Subtracting the two constraint terms gives (37). ■

A.6 Proposition 6

Proof. Denote $\sigma_A^{(1),*}$ as type A investors' optimal exposure to risk from sector 1, i.e., the first element of their risk exposure $\boldsymbol{\sigma}_A^*$. Using the market clearing condition for the risk exposure,

$$w\sigma_A^{(1),*} + (1-w)\sigma_B^{(1)} = z\sigma_{R,1}^{(1)} + (1-z)\sigma_{R,2}^{(1)},$$

express $\sigma_B^{(1)}$ in terms of $\sigma_A^{(1),*}$. Then plug it into the expression for the optimal risk exposure of a type B investor from (32),

$$\sigma_B^{(1)} = \frac{1}{\gamma_B}\eta^{(1)} + \frac{1}{\gamma_B}\frac{1-\gamma_B}{1-\psi}\sigma_{\xi_B}^{(1)},$$

substituting for $\eta^{(1)}$ using the expression for $\sigma_A^{(1),*}$,

$$\sigma_A^{(1),*} = \frac{1}{\gamma_A}\eta^{(1)} + \frac{1}{\gamma_A}\frac{1-\gamma_A}{1-\psi}\sigma_{\xi_A}^{(1)}.$$

Then I can compute $\sigma_A^{(1),*}$ as follows:

$$\sigma_A^{(1),*} = \frac{zR_w + R_wQ_x x(1-x) - R_x(Q_w + \frac{1}{w})x(1-x) - z\frac{\gamma_B}{w(1-w)} - \frac{\gamma_B}{w(1-w)}x(1-x)Q_x}{(\gamma_B-\gamma_A)Q_w - \frac{\gamma_B}{1-w} - \frac{\gamma_A}{w} + R_w}\sigma_1, \tag{80}$$

where

$$R_x = \frac{1-\gamma_A}{1-\psi}\frac{\xi_{A,x}}{\xi_A} - \frac{1-\gamma_B}{1-\psi}\frac{\xi_{B,x}}{\xi_B}; \quad R_w = \frac{1-\gamma_A}{1-\psi}\frac{\xi_{A,w}}{\xi_A} - \frac{1-\gamma_B}{1-\psi}\frac{\xi_{B,w}}{\xi_B};$$
$$Q_x = z\frac{q_{1,x}}{q_1} + (1-z)\frac{q_{2,x}}{q_2}; \qquad Q_w = z\frac{q_{1,w}}{q_1} + (1-z)\frac{q_{2,w}}{q_2}.$$

In a similar manner, I can compute $\sigma_A^{(2),*}$, the second element of $\boldsymbol{\sigma}_A^*$, as follows:

$$\sigma_A^{(2),*} = \frac{(1-z)R_w - R_wQ_x x(1-x) + R_x(Q_w + \frac{1}{w})x(1-x) - \frac{(1-z)\gamma_B}{w(1-w)} + \frac{\gamma_B}{w(1-w)}x(1-x)Q_x}{(\gamma_B-\gamma_A)Q_w - \frac{\gamma_B}{1-w} - \frac{\gamma_A}{w} + R_w}\sigma_2. \tag{81}$$

The risk exposure of type B investors can be computed using the market clearing condition for risk exposure. ■

A.7 Proposition 7

Proof. Plugging the (frictionless) optimal risk exposure (32) for type A and B investors into the market clearing conditions of risk exposure, I derive the expression for the vector of market

prices of risk

$$\begin{aligned}\boldsymbol{\eta}_t = \gamma_t \left[z_t \boldsymbol{\sigma}_{R_1,t} + (1-z_t)\boldsymbol{\sigma}_{R_2,t} \right] \\ - \gamma_t \left[w_t \left(\frac{1}{\gamma_A} \right) \left(\frac{1-\gamma_A}{1-\psi} \right) \boldsymbol{\sigma}_{\xi_A,t} + (1-w_t) \left(\frac{1}{\gamma_B} \right) \left(\frac{1-\gamma_B}{1-\psi} \right) \boldsymbol{\sigma}_{\xi_B,t} \right],\end{aligned} \tag{82}$$

where γ_t denotes aggregate risk aversion and is given by

$$\gamma_t = \left(\frac{w_t}{\gamma_A} + \frac{1-w_t}{\gamma_B} \right)^{-1}. \tag{83}$$

■

A.8 Proposition 8

Proof. Plugging the optimal consumption-wealth ratio (31) into the HJB (30), I obtain

$$c_{i,t} = \psi\rho + \mu_{\xi_i,t} + (1-\psi)(r_t + \Phi_{i,t}), \tag{84}$$

where

$$\begin{aligned}\Phi_{i,t} = \boldsymbol{\sigma}'_{i,t}\boldsymbol{\eta}_t - \frac{\gamma_i}{2}\boldsymbol{\sigma}'_{i,t}\boldsymbol{\sigma}_{i,t} \\ + \frac{1}{2}\frac{1}{1-\psi}\left(\frac{1-\gamma_i}{1-\psi} - 1 \right)\boldsymbol{\sigma}'_{\xi_i,t}\boldsymbol{\sigma}_{\xi_i,t} + \frac{1-\gamma_i}{1-\psi}\boldsymbol{\sigma}'_{i,t}\boldsymbol{\sigma}_{\xi_i,t}.\end{aligned} \tag{85}$$

Plugging the above equation into the goods market clearing condition, I obtain the expression for the interest rate

$$\psi\rho + \mu_t + (1-\psi)(r_t + \Phi_t) = z_t \frac{A_1 - \iota_1(g_{1,t})}{q_{1,t}} + (1-z_t)\frac{A_2 - \iota_2(g_{2,t})}{q_{2,t}}, \tag{86}$$

where $\mu_t = w_t\mu_{\xi_A,t} + (1-w_t)\mu_{\xi_B,t}$, and $\Phi_t = w_t\Phi_{A,t} + (1-w_t)\Phi_{B,t}$. ■

A.9 Lemma 1

Proof. The benchmark economy is essentially non-stochastic, up to scale. The equilibrium conditions are:

$$\frac{1}{1-1/\psi}(c_i-\rho)+(\sigma_i\eta+r-c_i)-\frac{\gamma_i}{2}\sigma_i^2=0,$$
$$\sigma_i=\frac{\eta}{\gamma_i},$$
$$\frac{A-\iota(g)}{q}+g-r-\eta\sigma=0,$$
$$\iota'(g)=q,$$
$$c=\frac{A-\iota(g)}{q},$$
$$\sigma_i=\sigma.$$

The market price of risk is

$$\eta^*=\gamma\sigma.$$

The interest rate is given by:

$$r^*=\rho+\psi^{-1}g^*-\left(1+\psi^{-1}\right)\frac{\gamma\sigma^2}{2}.$$

The consumption-wealth ratios, i.e., the dividend yield on aggregate wealth, are

$$c_A^*=c_B^*=\rho-\left(1-\psi^{-1}\right)\left(g^*-\frac{\gamma\sigma^2}{2}\right).$$

The growth rate g^* and price of capital q^* satisfy

$$\frac{A-\iota(g^*)}{q^*}=\rho-\left(1-\psi^{-1}\right)\left(g^*-\frac{\gamma\sigma^2}{2}\right),$$
$$\iota'(g^*)=q^*.$$

■

A.10 Lemma 2

This economy is the same as the one in Eberly and Wang (2025), except that it is solved in decentralized equilibrium. Eberly and Wang (2025) solve the social planner's problem, which

is equivalent to the decentralized equilibrium in this case.

Assume $\gamma_A = \gamma_B = \gamma$ and $\kappa = 0$, i.e., there is a representative investor. In addition, assume

$$\frac{dK_{1,t}}{K_{1,t}} = g_{1,t}dt + \sigma\varepsilon dB_{1,t} + \sigma(1-\varepsilon)dB_{2,t},$$
$$\frac{dK_{2,t}}{K_{2,t}} = g_{2,t}dt + \sigma dB_{2,t}.$$

The two sectors are assumed to be symmetric for expositional simplicity: productivity A, depreciation rate δ, and adjustment cost (ϕ,θ) are assumed to be the same across the two sectors.

Proof. We consider a first-order asymptotic expansion around the point $\varepsilon = 0$, i.e., for each variable z, we want to find the functions $(z_0(x), z_1(x))$ such that $z(x;\varepsilon) = z_0(x) + z_1(x)\varepsilon + O(\varepsilon^2)$, in which x, the capital share of sector 1, is the state variable.

The zeroth-order coefficients are derived in the benchmark economy, meaning that the economy in Lemma 1 corresponds to the case of $\varepsilon = 0$, for which an explicit solution exists. For the case where $\varepsilon > 0$, the equilibrium objects can be approximated by a Taylor expansion around $\varepsilon = 0$. For example, the price of capital in sector 1 can be approximated as

$$q_1(x;\varepsilon) = q^* + q_{1,\varepsilon}(x;0)\varepsilon + O(\varepsilon^2),$$

where q^* is the price of capital in the economy in Lemma 1. By computing the derivatives with respect to ε, $q_{1,\varepsilon}(x;0)$, we can obtain explicit expressions for the equilibrium objects when the extent of heterogeneity is small. The other equilibrium objects can be approximated similarly.

Using the two expressions above for the laws of motion of capital, the diffusion of x to the first shock from (15) can be written as

$$\sigma_{x,1} = x(1-x)\varepsilon\sigma,$$

and the diffusion of x to the second shock can be written as

$$\sigma_{x,2} = -x(1-x)\varepsilon\sigma.$$

Since the derivative of $q(x;0)$ (zeroth-order coefficient) with respect to x is zero, the derivative of $q(x;\varepsilon)$ with respect to x, namely $q_x(x;\varepsilon)$, is first-order in ε. Similarly, the derivative of the consumption-wealth ratio $\xi_i(x;\varepsilon)$ with respect to x, namely $\xi_{i,x}(x;\varepsilon)$, is also first-order in ε. Meanwhile, the diffusion terms of x are also first-order in ε as can be seen from above. Thus $\sigma_{q,1}, \sigma_{q,2}$ and $\sigma_{\xi_{i,1}}, \sigma_{\xi_{i,2}}$ are of order $O(\varepsilon^2)$ (see the proofs of Propositions 3 and 4 for ex-

pressions).

Combining the above with the market prices of risk from (42), we can write the market prices of risk as

$$\eta_1(x;\varepsilon) = \gamma\sigma z\varepsilon + O(\varepsilon^2),$$
$$\eta_2(x;\varepsilon) = \gamma(\sigma - \sigma z\varepsilon) + O(\varepsilon^2).$$

The risk exposure of investors from (32) can be written as

$$\sigma_{i,1} = \sigma z\varepsilon + O(\varepsilon^2),$$
$$\sigma_{i,2} = \sigma - \sigma z\varepsilon + O(\varepsilon^2).$$

From the optimal consumption-wealth ratio (84), we can write it as

$$c_i(x;\varepsilon) = \psi\rho + (1-\psi)(r_t + \frac{\gamma_i}{2}\sigma^2 - \gamma_i\sigma^2 z\varepsilon) + O(\varepsilon^2).$$

where we use $\mu_{\xi_i,t} = O(\varepsilon^2)$, and $\Phi_{i,t} = \frac{\gamma_i}{2}\sigma^2 - \gamma_i\sigma^2 z\varepsilon + O(\varepsilon^2)$.

Write the drift of x from (15) as

$$\mu_x = x(1-x)\left[g_1 - g_2\right] + 2x^3\varepsilon^2\sigma^2 - 2x^2\varepsilon^2\sigma^2 - x^2\varepsilon\sigma^2 + x\varepsilon\sigma^2$$

Since the derivative $c_{i,x}(x;\varepsilon)$ is first-order in ε, the drift terms of c and q are second-order in ε.

Thus the firms' HJB equations can be written as

$$\frac{A - \iota(g_n)}{q_n} + g_n + - r_t - \eta_1\sigma_{n,1} - \eta_2\sigma_{n,2} = 0.$$

where we use $\mu_{q,n} = O(\varepsilon^2)$, and $\sigma_{q_n,1}, \sigma_{q_n,2} = O(\varepsilon^2)$.

Thus we have

$$\frac{A - \iota(g_1)}{q_1} = r_t - g_1 + \gamma\sigma^2 - \gamma\sigma^2 z\varepsilon - \gamma\sigma^2\varepsilon + O(\varepsilon^2),$$
$$\frac{A - \iota(g_2)}{q_2} = r_t - g_2 + \gamma\sigma^2 - \gamma\sigma^2 z\varepsilon + O(\varepsilon^2).$$

Using the HJB equations for investors and firms and the market clearing condition for

consumption goods, we obtain the interest rate

$$\begin{aligned}
r_t &= \rho - (1-\psi^{-1})(\frac{\gamma}{2}\sigma^2 - \gamma\sigma^2 z\varepsilon) - \psi^{-1}\left[\gamma\sigma^2 - 2\gamma\sigma^2 z\varepsilon - zg_1 - (1-z)g_2\right] \\
&= \rho + \psi^{-1}g^* - (1+\psi^{-1})\frac{\gamma}{2}\sigma^2 \\
&+ (1+\psi^{-1})\gamma\sigma^2 z\varepsilon + \psi^{-1}\left[z(g_1 - g^*) + (1-z)(g_2 - g^*)\right] + O(\varepsilon^2), \\
&= r^* \\
&+ (1+\psi^{-1})\gamma\sigma^2 z\varepsilon + \psi^{-1}\left[z(g_1 - g^*) + (1-z)(g_2 - g^*)\right] + O(\varepsilon^2).
\end{aligned}$$

Following the first-order asymptotic expansion, write the price of capital as

$$\begin{aligned}
q_1 &= q^* + q_{1,1}(x)\varepsilon + O(\varepsilon^2), \\
q_2 &= q^* + q_{2,1}(x)\varepsilon + O(\varepsilon^2),
\end{aligned}$$

where q^* is the equilibrium price of capital from Lemma 1, and $q_{1,1}$ denotes the first-order coefficient of q_1 with respect to ε. The same expansion applies to $q_{2,1}$.

From (21), $g = \frac{q-\phi}{\theta} - \delta$. Then the growth rate can be written as

$$\begin{aligned}
g_1 &= g^* + \frac{q_{1,1}}{\theta}\varepsilon + O(\varepsilon^2), \\
g_2 &= g^* + \frac{q_{2,1}}{\theta}\varepsilon + O(\varepsilon^2),
\end{aligned}$$

where $g^* = \frac{q^*-\phi}{\theta} - \delta$ is the equilibrium growth rate from Lemma 1.

Similarly, (21) gives $\iota = \frac{q^2-\phi^2}{2\theta}$, so the investment rate can be written as

$$\begin{aligned}
\iota_1 &= \frac{(q^*)^2 + 2q^* q_{1,1}\varepsilon - \phi^2}{2\theta} + O(\varepsilon^2), \\
\iota_2 &= \frac{(q^*)^2 + 2q^* q_{2,1}\varepsilon - \phi^2}{2\theta} + O(\varepsilon^2).
\end{aligned}$$

The variable z is defined in (22), which can be written as

$$z = \left(1 + \frac{q_{2,t}}{q_{1,t}}\frac{1-x_t}{x_t}\right)^{-1}.$$

Substituting the expressions for r_t, the growth rate, the price of capital, and the investment rate into the firms' HJB equations, we can solve for $(q_{1,1}, q_{2,1})$ jointly, which gives the first-order

asymptotic expansion of the equilibrium objects. The two equations are:

$$0 = c^* q_{1,1}\varepsilon + q^*\psi^{-1}\gamma\sigma^2 z\varepsilon + \psi^{-1}q^*\left[z\frac{q_{1,1}}{\theta} + (1-z)\frac{q_{2,1}}{\theta}\right]\varepsilon - q^*\gamma\sigma^2\varepsilon,$$
$$0 = c^* q_{2,1}\varepsilon + q^*\psi^{-1}\gamma\sigma^2 z\varepsilon + \psi^{-1}q^*\left[z\frac{q_{1,1}}{\theta} + (1-z)\frac{q_{2,1}}{\theta}\right]\varepsilon.$$

where $c^* = r^* - g^* + \gamma\sigma^2$ is the consumption-wealth ratio from Lemma 1.

The solution to the above pair of equations gives

$$q_{1,1} = \frac{q^*\gamma\sigma^2(-c^* x\theta + c^*\theta\psi - q^* x + q^*)}{c^*(c^*\theta\psi + q^*)},$$
$$q_{2,1} = \frac{q^* x\gamma\theta\sigma^2(-c^*\theta - q^*)}{c^*(c^*\theta\psi + q^*)}.$$

The derivative of the price of capital with respect to x, which is the same for both sectors, can be written as

$$q_x(x;\varepsilon) = \frac{q^*\gamma\sigma^2(-c^*\theta - q^*)}{c^*(c^*\theta\psi + q^*)}\varepsilon,$$

which is negative in the reasonable calibration.

To complete the proof, the drift of x can be written as

$$\mu_x = x(1-x)\frac{q^*\gamma\sigma^2}{c^*\theta}\varepsilon - x^2\varepsilon\sigma^2 + x\varepsilon\sigma^2 + O(\varepsilon^2).$$

■

A.11 Lemma 3

This economy is a standard frictionless heterogeneous-agent asset pricing model in a production economy.

Proof. We consider a first-order asymptotic expansion around the point $\epsilon = 0$, i.e., for each variable z, we want to find the functions $(z_0(w), z_1(w))$ such that $z(w;\epsilon) = z_0(w) + z_1(w)\epsilon + O(\epsilon^2)$. The zeroth-order coefficients are derived in the benchmark economy (see Lemma 1).

Let $\gamma_A = \gamma(1-\Delta\gamma)$, $\gamma_B = \gamma(1+\Delta\gamma)$, $\Delta\gamma > 0$, and

$$\Delta\gamma = \delta_\gamma\epsilon, \quad \kappa = \delta_\kappa\epsilon.$$

Aggregate risk aversion from (43) can be written as

$$\gamma(w;\epsilon) = \gamma\left(\frac{1-w}{1+\Delta\gamma} + \frac{w}{1-\Delta\gamma}\right)^{-1} = \gamma - (2w-1)\gamma\Delta\gamma + O(\epsilon^2).$$

In equilibrium, one of the FOCs gives $c_i = \xi_i$, so the proof focuses on characterizing c_i.

Since the derivative of $q(w;0)$ (zeroth-order coefficient) with respect to w is zero, the derivative of $q(w;\epsilon)$ with respect to w, namely $q_w(w;\epsilon)$, is first-order in ϵ. Similarly, the derivative of the consumption-wealth ratio $c_i(w;\epsilon)$ with respect to w, namely $c_{i,w}(w;\epsilon)$, is also first-order in ϵ.

Meanwhile, the diffusion of w, σ_w is also first-order in ϵ because it is equal to zero at $\epsilon = 0$. Thus σ_q and σ_{ξ_i} are both of order $O(\epsilon^2)$ (see the proofs of Propositions 3 and 4 for expressions).

Then it follows that the market price of risk from (42) can be written as

$$\eta_t(w;\epsilon) = \gamma\sigma - \gamma\sigma(2w-1)\Delta\gamma + O(\epsilon^2).$$

Note that at $\epsilon = 0$, $\eta(w;0) = \gamma\sigma$, which is the market price of risk η^* from the benchmark economy in Lemma 1.

Then the risk exposure of investors from (32) can be written as

$$\begin{aligned}\sigma_A(w;\epsilon) &= \sigma + 2(1-w)\sigma\Delta\gamma + O(\epsilon^2),\\ \sigma_B(w;\epsilon) &= \sigma - 2w\sigma\Delta\gamma + O(\epsilon^2).\end{aligned}$$

Similarly, note that at $\epsilon = 0$, $\sigma_A(w;0) = \sigma_B(w;0) = \sigma$, which is the risk exposure from the benchmark economy in Lemma 1.

Using the above two expressions, the diffusion of w from (14) can be written as

$$\sigma_w(w;\epsilon) = w(1-w)2\sigma\Delta\gamma + O(\epsilon^2).$$

The interest rate, from the firm's HJB equation, can be written as

$$r_t = \frac{A - \iota(g_t)}{q_t} + g_t + \mu_q + \sigma_q\sigma - \eta_t(\sigma_q + \sigma)$$

Using the market clearing condition for consumption goods, we can write the interest rate as

$$\begin{aligned}r_t &= [wc_A + (1-w)c_B] + g_t + \mu_q + \sigma_q\sigma - \eta_t(\sigma_q + \sigma)\\ &= \psi\rho + \mu_t + (1-\psi)(r_t + \Phi_t) + g_t + \mu_q + \sigma_q\sigma - \eta_t(\sigma_q + \sigma)\end{aligned}$$

where the second line uses the optimal conditions for the consumption-wealth ratio (84) for investors A and B. Collecting the terms in r_t on one side gives

$$r_t = \rho + \psi^{-1}[\mu_t + g_t + \mu_q + \sigma_q\sigma - \eta_t(\sigma_q + \sigma)] - (1-\psi^{-1})\Phi_t.$$

where

$$\begin{aligned}
\mu_t &= w\mu_{\xi_A,t} + (1-w)\mu_{\xi_B,t} = O(\epsilon^2),\\
\Phi_{A,t} &= \frac{\gamma\sigma^2}{2} - \gamma\sigma^2(2w-1)\Delta\gamma + \frac{\gamma\sigma^2}{2}\Delta\gamma + O(\epsilon^2)\\
\Phi_{B,t} &= \frac{\gamma\sigma^2}{2} - \gamma\sigma^2(2w-1)\Delta\gamma - \frac{\gamma\sigma^2}{2}\Delta\gamma + O(\epsilon^2)\\
\Phi_t &= w\Phi_{A,t} + (1-w)\Phi_{B,t}
\end{aligned}$$

The first equation above follows because μ_w is zero at $\epsilon = 0$, which means $\mu_w(w;\epsilon)$ is first-order in ϵ, and the derivative $c_{i,w}(w;\epsilon)$ is first-order in ϵ. The drift terms of c and q are second-order in ϵ. Thus,

$$\begin{aligned}
r_t(w;\epsilon) = \rho &+ \psi^{-1}g^* - (1+\psi^{-1})\frac{\gamma\sigma^2}{2}\\
&+ \psi^{-1}(g_t - g^*) - \psi^{-1}\sigma(\eta_t - \eta^*)\\
&- (1-\psi^{-1})[w(\Phi_{A,t} - \Phi^*_{A,t}) + (1-w)(\Phi_{B,t} - \Phi^*_{B,t})] + O(\epsilon^2)\\
&= r^*\\
&+ \psi^{-1}(g_t - g^*) - \psi^{-1}\sigma(\eta_t - \eta^*)\\
&- (1-\psi^{-1})[w(\Phi_{A,t} - \Phi^*_{A,t}) + (1-w)(\Phi_{B,t} - \Phi^*_{B,t})] + O(\epsilon^2)
\end{aligned}$$

Following the first-order asymptotic expansion, write the price of capital as

$$q = q^* + q_1(w)\epsilon + O(\epsilon^2).$$

where q^* is the equilibrium price of capital from Lemma 1, and q_1 denotes the first-order coefficient of q with respect to ϵ.

From (21), $g = \frac{q-\phi}{\theta} - \delta$. Then the growth rate can be written as

$$g = g^* + \frac{q_1}{\theta}\epsilon + O(\epsilon^2)$$

where $g^* = \frac{q^*-\phi}{\theta} - \delta$ is the equilibrium growth rate from Lemma 1.

Thus, the interest rate can be written as

$$r_t = r^* + \psi^{-1}\frac{q_1}{\theta}\epsilon + \frac{\gamma}{2}\sigma^2(1+\psi^{-1})(2w-1)\Delta\gamma + O(\epsilon^2)$$

After substituting the expression for r_t, the consumption-wealth ratios again from (84) can be written as

$$c_A = c^* + (1-\psi)\psi^{-1}\frac{q_1}{\theta}\epsilon + (1-\psi)[\frac{\gamma}{2}\sigma^2(\psi^{-1}-1)(2w-1) + \frac{\gamma\sigma^2}{2}]\Delta\gamma + O(\epsilon^2)$$
$$c_B = c^* + (1-\psi)\psi^{-1}\frac{q_1}{\theta}\epsilon + (1-\psi)[\frac{\gamma}{2}\sigma^2(\psi^{-1}-1)(2w-1) - \frac{\gamma\sigma^2}{2}]\Delta\gamma + O(\epsilon^2)$$

Substituting these expressions back into the market clearing condition for consumption goods, we obtain an expression for $q_1(w)$ from

$$c^* + (\psi^{-1}-1)\frac{q_1}{\theta}\epsilon + (\psi^{-1}-1)\frac{\gamma}{2}\sigma^2(2w-1)\Delta\gamma + O(\epsilon^2) = \frac{A-\iota(g(q))}{q}$$

Using Lemma 1, $c^* = \frac{A-\iota(g(q^*))}{q^*}$, and $\iota = \frac{q^2-\phi^2}{2\theta}$, the expression for $q_1(w)$ can be simplified to

$$q_1(w) = \frac{(\psi-1)\theta q^*}{q^* + \psi\theta c^*}(2w-1)\frac{\gamma}{2}\sigma^2\delta_\gamma$$

Now all equilibrium objects can be expressed as functions of w and ϵ. For example, the growth rate of the economy can be written as

$$g(w;\epsilon) = g^* + \frac{(\psi-1)q^*}{q^* + \psi\theta c^*}(2w-1)\frac{\gamma}{2}\sigma^2\Delta\gamma + O(\epsilon^2)$$

Note that when $\psi > 1$, $q_1(w)$ is increasing in w, i.e., the growth rate is increasing in w.

To complete the derivation, we have expressions for the drift term of w from (14). The second component can be written as

$$\begin{aligned}\mu_w(w;\epsilon) &= [(\sigma_A - \sigma_B)\eta + c_B - c_A] - (\sigma_A - \sigma_B)\sigma_R \\ &= (1 + \psi - \frac{2}{\gamma})\gamma\sigma^2\Delta\gamma + O(\epsilon^2),\end{aligned}$$

and the first component can be written as $\kappa(\bar{u} - w)$. ■

A.12 Lemma 4

This economy is a heterogeneous-agent asset pricing model in a production economy with financial frictions. Lemma 3 shows that the frictionless optimal risk exposure σ_A, given by

$$\sigma_A^*(w;\epsilon) = \sigma + 2(1-w)\sigma\Delta\gamma + O(\epsilon^2),$$

is linearly decreasing in w. This implies that as the wealth share of type A investors decreases, their optimal risk exposure increases. Thus, financial constraints will bind for type A investors in equilibrium when their wealth share is sufficiently low.

Building on Lemma 3, we consider the case where the financial constraint is binding, i.e., the investor's risk exposure is constrained. We derive the equilibrium under this binding constraint. Lemma 3 characterizes the equilibrium when the constraints are slack. This lemma characterizes the equilibrium when the constraint binds for type A investors.

Proof. Consider $\sigma_A = \bar{\zeta}$. Following the same arguments that σ_q and σ_{ξ_i} are both of order $O(\epsilon^2)$, as in Lemma 3, we can solve for the risk exposure of type B investors under the binding constraint. Using the market clearing condition for risk exposure, the risk exposure of type B investors can be written as

$$\sigma_B(w;\epsilon) = \frac{\sigma - w\bar{\zeta}}{1-w} + O(\epsilon^2).$$

The market price of risk can be written as

$$\eta_t(w;\epsilon) = \gamma_B \frac{\sigma - w\bar{\zeta}}{1-w} + O(\epsilon^2).$$

We can write the following objects as

$$\begin{aligned}
\Phi_{A,t} &= \gamma_B \frac{(\sigma - w\bar{\zeta})\bar{\zeta}}{1-w} - \frac{\gamma_A \bar{\zeta}^2}{2} + O(\epsilon^2) \\
\Phi_{B,t} &= \frac{\gamma_B}{2}\left(\frac{(\sigma - w\bar{\zeta})}{1-w}\right)^2 + O(\epsilon^2) \\
\Phi_t &= w\Phi_{A,t} + (1-w)\Phi_{B,t}
\end{aligned}$$

Using the market clearing condition for goods, the interest rate can be expressed as

$$r_t = \rho + \psi^{-1}[\mu_t + g_t + \mu_q + \sigma_q\sigma - \eta_t(\sigma_q + \sigma)] - (1-\psi^{-1})\Phi_t.$$

Using the same arguments as in Lemma 3, we can write

$$\begin{aligned} r_t(w;\epsilon) &= \rho + \psi^{-1}(g_t - \eta_t\sigma) - (1-\psi^{-1})\Phi_t + O(\epsilon^2) \\ &= \rho + \psi^{-1}g^* - (1+\psi^{-1})\frac{\gamma\sigma^2}{2} \\ &+ \psi^{-1}\frac{q_1}{\theta}\epsilon - \psi^{-1}\eta_t\sigma - (1-\psi^{-1})\Phi_t + (1+\psi^{-1})\frac{\gamma\sigma^2}{2} + O(\epsilon^2) \\ &= r^* \\ &+ \psi^{-1}\frac{q_1}{\theta}\epsilon - \psi^{-1}\eta_t\sigma - (1-\psi^{-1})\Phi_t + (1+\psi^{-1})\frac{\gamma\sigma^2}{2} + O(\epsilon^2) \end{aligned}$$

Putting the above expression for the interest rate back into the firm's HJB equation, we can solve for $q_1(w)$ as

$$q_1(w) = q^*\frac{2w^2\bar{\zeta}^2\psi - 2w^2\bar{\zeta}^2 - w\bar{\zeta}^2\psi + w\bar{\zeta}^2 - 2w\bar{\zeta}\sigma\psi + 2w\bar{\zeta}\sigma + \psi\sigma^2 - \sigma^2}{-g^*w\theta\psi + g^*\theta\psi + q^*w - q^* + r^*w\theta\psi - r^*\theta\psi + w\gamma\bar{\zeta}\theta\sigma\psi - \gamma\theta\psi\sigma^2}\theta\frac{\gamma}{2}\delta_\gamma.$$

Other equilibrium objects, including the drift term of w, can be expressed accordingly, as in Lemma 3. ■

A.13 Consumption-equivalent welfare

Fix the initial state (K, x, w), with $0 < w < 1$, and let $r \in \{u, \zeta\}$ denote the unconstrained economy or the economy with risk limit ζ. Write investor i's homothetic value function in terms of a positive certainty equivalent H_i^r and its value per unit of wealth $B_i^r(x,w)$,

$$V_i^r = \frac{(H_i^r)^{1-\gamma_i}}{1-\gamma_i}, \qquad H_i^r = B_i^r W_i^r.$$

The optimal consumption rule recovers B_i^r from the consumption-wealth ratio $\xi_i^r = C_i^r/W_i^r$. Equating marginal utility of consumption to the marginal value of wealth and canceling their common factor $(H_i^r)^{-\gamma_i}$ gives

$$(C_i^r/H_i^r)^{-1/\psi} = B_i^r, \qquad \xi_i^r = (B_i^r)^{1-\psi}, \qquad B_i^r = (\xi_i^r)^{1/(1-\psi)}.$$

Substituting into V_i^r recovers (28). Any common multiplicative normalization of the certainty equivalent cancels in the comparison across regimes.

By homotheticity, scaling the entire unconstrained consumption stream by $1+\Delta_i^\zeta$ scales its certainty equivalent by the same factor. To match constrained welfare, the required scale

factor therefore satisfies

$$1+\Delta_i^{\zeta}=\frac{H_i^{\zeta}}{H_i^{u}}=\frac{W_i^{\zeta}}{W_i^{u}}\left(\frac{\xi_i^{\zeta}}{\xi_i^{u}}\right)^{1/(1-\psi)}.$$

The initial wealth ratio follows from the valuation of the two capital stocks. In regime r, their combined value is $q^r K$, where $q^r = x q_1^r + (1-x) q_2^r$. Intermediaries own $w q^r K$ and households own $(1-w) q^r K$. Since the comparison fixes the physical capital stocks and each type's wealth share, $W_i^{\zeta}/W_i^{u} = q^{\zeta}/q^{u}$ for both types. This gives (48), or equivalently,

$$\log(1+\Delta_i^{\zeta}) = \log(q^{\zeta}/q^{u}) + \frac{1}{1-\psi}\log(\xi_i^{\zeta}/\xi_i^{u}).$$

The first term is common to both types. The second captures the change in the certainty equivalent per unit of wealth for type i. Both terms adjust in equilibrium. The welfare figures report $100\Delta_i^{\zeta}$, so a value of five means that raising the investor's unconstrained consumption by five percent at every date and in every state delivers the same lifetime utility as the constrained regime.

B Numerical solution

The computation of equilibrium is reduced to the solution of a system of partial differential equations (PDEs) involving $(\xi_A(x,w), \xi_B(x,w), q_1(x,w), q_2(x,w))$.

The following procedure shows how to obtain a set of conditions involving (ξ_A, ξ_B, q_1, q_2) and their derivatives:

i. Using the optimal investment rate for firms in sector n from (21), I can express ι_n and g_n as functions of q_n.

ii. Using the expression for g_n from above, I can compute μ_x, the drift of the state variable x from (15). Note that the diffusion terms of x are exogenously given.

iii. Compute the variable z as a function of (q_1, q_2) from (22).

iv. Then I solve for the dynamics of the state variable w, in particular the diffusion terms of w. Using the solutions for type A investors' risk exposures (38) from Proposition 6, and considering the margin constraints, I have

$$\sigma_A^{(1)} = \min(\sigma_A^{(1),*}, \bar{\zeta}^{(1)}), \quad \sigma_A^{(2)} = \min(\sigma_A^{(2),*}, \bar{\zeta}^{(2)}).$$

v. Using the market clearing conditions for the risk exposures again, I can compute $\sigma_B^{(1)}$ and $\sigma_B^{(2)}$ as follows:

$$\sigma_B^{(1)} = \frac{z\sigma_1 + Q_x x(1-x)\sigma_1 + [Q_w w(1-w) - w]\sigma_A^{(1)}}{Q_w w(1-w) + 1 - w},$$
$$\sigma_B^{(2)} = \frac{(1-z)\sigma_2 - Q_x x(1-x)\sigma_2 + [Q_w w(1-w) - w]\sigma_A^{(2)}}{Q_w w(1-w) + 1 - w}.$$

vi. Compute $\sigma_w^{(1)}$ and $\sigma_w^{(2)}$ using (14).

vii. Applying Itô's lemma and using the diffusion terms of x and w, compute the following:

$$\sigma_{q,n}^{(1)} = \frac{q_{n,x}}{q_n}\sigma_x^{(1)} + \frac{q_{n,w}}{q_n} w(1-w)\sigma_w^{(1)}; \quad \sigma_{q,n}^{(2)} = \frac{q_{n,x}}{q_n}\sigma_x^{(2)} + \frac{q_{n,w}}{q_n} w(1-w)\sigma_w^{(2)};$$
$$\sigma_{\xi,i}^{(1)} = \frac{\xi_{i,x}}{\xi_i}\sigma_x^{(1)} + \frac{\xi_{i,w}}{\xi_i} w(1-w)\sigma_w^{(1)}; \quad \sigma_{\xi,i}^{(2)} = \frac{\xi_{i,x}}{\xi_i}\sigma_x^{(2)} + \frac{\xi_{i,w}}{\xi_i} w(1-w)\sigma_w^{(2)}.$$

viii. Compute $(\eta^{(1)}, \eta^{(2)})$ using the expression for optimal risk exposures for type B investors from (32):

$$\eta^{(1)} = \gamma_B \sigma_B^{(1)} - \frac{1-\gamma_B}{1-\psi}\sigma_{\xi,B}^{(1)}, \quad \eta^{(2)} = \gamma_B \sigma_B^{(2)} - \frac{1-\gamma_B}{1-\psi}\sigma_{\xi,B}^{(2)}.$$

ix. Compute $(\boldsymbol{\sigma}_{R,1}, \boldsymbol{\sigma}_{R,2})$ by definition (26), which also defines $\boldsymbol{\Sigma}_R$.

x. Compute μ_w from (14), using optimal consumption-wealth ratios $c_i = \xi_i$ from (31) and the expressions above.

xi. Applying Itô's lemma, compute $\mu_{q,n}$ for $n \in \{1,2\}$ and $\mu_{\xi,i}$ for $i \in \{A,B\}$ as follows:

$$\begin{aligned}
\mu_{q,n} &= \frac{q_{n,x}}{q_n}\mu_x + \frac{q_{n,w}}{q_n}\left[\kappa(\bar{\mu} - w) + w(1-w)\mu_w\right] \\
&\quad + \frac{1}{2}\frac{q_{n,xx}}{q_n}(\sigma_{x,1}^2 + \sigma_{x,2}^2) + \frac{1}{2}\frac{q_{n,ww}}{q_n}w^2(1-w)^2(\sigma_{w,1}^2 + \sigma_{w,2}^2) \\
&\quad + \frac{q_{n,xw}}{q_n}w(1-w)(\sigma_{x,1}\sigma_{w,1} + \sigma_{x,2}\sigma_{w,2}); \\
\mu_{\xi,i} &= \frac{\xi_{i,x}}{\xi_i}\mu_x + \frac{\xi_{i,w}}{\xi_i}\left[\kappa(\bar{\mu} - w) + w(1-w)\mu_w\right] \\
&\quad + \frac{1}{2}\frac{\xi_{i,xx}}{\xi_i}(\sigma_{x,1}^2 + \sigma_{x,2}^2) + \frac{1}{2}\frac{\xi_{i,ww}}{\xi_i}w^2(1-w)^2(\sigma_{w,1}^2 + \sigma_{w,2}^2) \\
&\quad + \frac{\xi_{i,xw}}{\xi_i}w(1-w)(\sigma_{x,1}\sigma_{w,1} + \sigma_{x,2}\sigma_{w,2}).
\end{aligned}$$

xii. Next I solve for the interest rate r. Plugging the optimal consumption-wealth ratio (31) into the investors' HJB (30), I obtain

$$c_{i,t} - \psi\rho - \mu_{\xi_i,t} - (1-\psi)(r_t + \Phi_{i,t}) = 0,$$

where

$$\begin{aligned}
\Phi_{i,t} &= \boldsymbol{\sigma}_{i,t}'\boldsymbol{\eta}_t - \frac{\gamma_i}{2}\boldsymbol{\sigma}_{i,t}'\boldsymbol{\sigma}_{i,t} \\
&\quad + \frac{1}{2}\frac{1}{1-\psi}\left(\frac{1-\gamma_i}{1-\psi} - 1\right)\boldsymbol{\sigma}_{\xi_i,t}'\boldsymbol{\sigma}_{\xi_i,t} + \frac{1-\gamma_i}{1-\psi}\boldsymbol{\sigma}_{i,t}'\boldsymbol{\sigma}_{\xi_i,t}.
\end{aligned}$$

Plugging the optimal investment rate (21) into the firms' HJB (20), I obtain

$$\frac{A_n - \iota_{n,t}}{q_{n,t}} + \Omega_{n,t} - r_t = 0,$$

where

$$\Omega_{n,t} = g_{n,t} + \mu_{q_{n,t}} + \boldsymbol{\sigma}_{q_{n,t}}'\boldsymbol{\sigma}_{n,t} - \boldsymbol{\eta}_t'(\boldsymbol{\sigma}_{q_{n,t}} + \boldsymbol{\sigma}_{n,t}).$$

Substituting these equations into the goods market clearing condition gives

$$wc_A + (1-w)c_B = z\frac{A_1 - \iota_1}{q_1} + (1-z)\frac{A_2 - \iota_2}{q_2}.$$

Rearranging the terms, I obtain the expression for the interest rate

$$r_t = \rho + \frac{1}{\psi}\mu_t + \left(\frac{1}{\psi} - 1\right)\Phi_t + \frac{1}{\psi}\Omega_t$$

where $\mu_t = w_t\mu_{\xi_A,t} + (1-w_t)\mu_{\xi_B,t}$, $\Phi_t = w_t\Phi_{A,t} + (1-w_t)\Phi_{B,t}$, and $\Omega_t = z\Omega_{1,t} + (1-z)\Omega_{2,t}$.

xiii. Plug all necessary equations into the HJB equations (20) and (30) to obtain the system of PDEs.

C Alternative economies

Although the model is analytically tractable, one may wonder how each segment of the model contributes to the equilibrium and if the full setup is necessary to capture the key mechanisms. In this section, I consider alternative economies that can provide intuition in this regard. I rely on the method of asymptotic expansion, which yields semi-closed-form solutions for the equilibrium objects of these economies. This sheds light on the role of each segment of the model before proceeding to numerical solutions.

The first (frictionless) economy features one representative sector and no heterogeneity in risk aversion among investors. The following lemma shows that up to scale, there are no fluctuations in the equilibrium objects, such as investment, consumption, and asset prices.

Lemma 1 *Consider an economy where $\gamma_A = \gamma_B = \gamma$ and the two sectors are identical. Under these assumptions, the structure of aggregate shocks simplifies to only one Brownian shock, with volatility σ, i.e.,*

$$\frac{dK_t}{K_t} = g_t\,dt + \sigma\,dB_t. \tag{87}$$

The following equilibrium objects can be derived.

i. The consumption-wealth ratio and risk exposure are given by

$$c_A^* = c_B^* = \rho - (1-\psi^{-1})\left(g^* - \frac{\gamma\sigma^2}{2}\right), \qquad \sigma_A^* = \sigma_B^* = \sigma. \tag{88}$$

ii. The investment rate and price of capital satisfy

$$\frac{A-\iota^*}{q^*} = \rho - \left(1-\psi^{-1}\right)\left(g^* - \frac{\gamma\sigma^2}{2}\right), \qquad g'(\iota^*) = \frac{1}{q^*}. \tag{89}$$

iii. The market price of risk and interest rate are given by

$$\eta^* = \gamma\sigma, \qquad r^* = \rho + \psi^{-1}g^* - (1+\psi^{-1})\frac{\gamma\sigma^2}{2}. \tag{90}$$

Proof. See Appendix A. ■

The above lemma shows that in the absence of heterogeneity in risk aversion, the one-sector economy is essentially deterministic. Output, consumption, and investment all move in proportion to the capital stock, which is subject to aggregate shocks. Since investment opportunities are constant in this case, investors maintain a constant portfolio weight of one over time. The market price of risk and the interest rate are both constant as well.

Consider next the second (frictionless) economy featuring two productive sectors and no heterogeneity in risk aversion among investors, à la Eberly and Wang (2025). A closed-form solution is unavailable in this case. Following Silva (2020), I characterize the behavior of the economy using an asymptotic expansion of the equilibrium around the economy in Lemma 1.[18]

Lemma 2 *Consider a parametric family of economies indexed by ε, where the two sectors are not identical. Assume $\gamma_A = \gamma_B = \gamma$. In addition, assume*

$$\frac{dK_{1,t}}{K_{1,t}} = g_{1,t}dt + \sigma\varepsilon dB_{1,t} + \sigma(1-\varepsilon)dB_{2,t}, \tag{91}$$

$$\frac{dK_{2,t}}{K_{2,t}} = g_{2,t}dt + \sigma dB_{2,t}, \tag{92}$$

and productivity A, depreciation rate δ, and adjustment cost (ϕ,θ) are assumed to be the same across the two sectors for expositional simplicity.

The equilibrium objects, as functions of x, can be obtained to first order in ε.

i. The price of capital in sector 1 is given by

$$q_1(x;\varepsilon) = q^* + \frac{q^*\gamma\sigma^2(-c^*x\theta + c^*\theta\psi - q^*x + q^*)}{c^*(c^*\theta\psi + q^*)}\varepsilon + O(\varepsilon^2). \tag{93}$$

ii. The market prices of risk are given by

$$\eta_1(x;\varepsilon) = \gamma\sigma z\varepsilon + O(\varepsilon^2), \qquad \eta_2(x;\varepsilon) = \gamma(\sigma - \sigma z\varepsilon) + O(\varepsilon^2). \tag{94}$$

where z is the weight of sector 1 in the market portfolio, i.e., $z = \left(1 + \frac{q_2}{q_1}\frac{1-x}{x}\right)^{-1}$.

All other equilibrium objects can be found in Appendix A.

Proof. See Appendix A. ■

The above lemma highlights that the introduction of a second sector leads to the equilibrium objects becoming functions of the sectoral composition of the economy, x. In this unconstrained representative-investor economy, sectoral investment implements the planner's allocation subject to the same capital adjustment costs.[19] The representative investor bears both shocks, so changes in sectoral risk affect investment without any redistribution of risk across investors.

[18] Each sector's investment rate is monotonically increasing in its Tobin's Q, so the two convey essentially the same information for the given adjustment cost function.

[19] Appendix D presents the planner's problem and the global solution.

Consider next the third (frictionless) economy featuring one sector and heterogeneous investors. The behavior of the economy can again be characterized by considering an asymptotic expansion of the equilibrium around the economy in Lemma 1, as no closed-form solution is available in this case either.

Lemma 3 *Consider a parametric family of economies indexed by ϵ, where the two sectors are identical. The structure of aggregate shocks boils down to only one Brownian shock, with volatility σ, as in Lemma 1. Let $\gamma_A = \gamma(1-\Delta\gamma)$, $\gamma_B = \gamma(1+\Delta\gamma)$, $\Delta\gamma \equiv \delta_\gamma \epsilon$, and $\delta_\gamma > 0$.*

The equilibrium objects, as functions of w, can be obtained to first order in ϵ.

i. Risk exposures

$$\sigma_A(w;\epsilon) = \sigma + 2(1-w)\sigma\Delta\gamma + O(\epsilon^2), \tag{95}$$

$$\sigma_B(w;\epsilon) = \sigma - 2w\sigma\Delta\gamma + O(\epsilon^2). \tag{96}$$

ii. Market price of risk

$$\eta(w;\epsilon) = \gamma\sigma - \gamma\sigma(2w-1)\Delta\gamma + O(\epsilon^2). \tag{97}$$

iii. The growth rate of capital

$$g(w;\epsilon) = g^* + \frac{(\psi-1)q^*}{q^* + \phi_1\psi c^*}\frac{\gamma\sigma^2}{2}(2w-1)\Delta\gamma + O(\epsilon^2). \tag{98}$$

All other equilibrium objects can be found in Appendix A.

Proof. See Appendix A. ■

The above lemma, in comparison to Lemma 1, showcases the role of heterogeneity in risk aversion in driving fluctuations in the equilibrium. In this one-sector economy, intermediaries have greater exposure to the single shock to first order. Subtracting the two exposures gives $\sigma_A(w;\epsilon) - \sigma_B(w;\epsilon) = 2\sigma\Delta\gamma + O(\epsilon^2)$ for $w \in (0,1)$. Financial intermediaries use leverage, and after a negative shock, their wealth share declines, leading to an increase in the market price of risk and, provided $\psi > 1$, a fall in investment and growth.[20] This mechanism, often referred to as the *balance-sheet channel* in the macro-finance literature (Di Tella, 2017), is a result of risk sharing among investors, which makes the intermediaries' wealth share, w_t, a state variable that drives time variation in risk premia.[21] There is no capital reallocation or

[20] The condition $\psi > 1$ guarantees that the price of the risky asset falls after a negative shock, as the increase in the risk premium dominates the decline in interest rates.

[21] Appendix D presents the global solution of this economy.

misallocation in this economy, as the investors can only trade one risky asset, and the sectoral composition of the economy is constant.

The fourth economy introduces the financial constraints in the form of (10) (applied to one risky asset) on top of the setup in Lemma 3.

Lemma 4 *Consider a parametric family of economies indexed by ϵ, where the two sectors are identical. The structure of aggregate shocks boils down to only one Brownian shock, with volatility σ, as in Lemmas 1 and 3. Let $\gamma_A = \gamma(1-\Delta\gamma)$, $\gamma_B = \gamma(1+\Delta\gamma)$, $\Delta\gamma \equiv \delta_\gamma \epsilon$, and $\delta_\gamma > 0$. Moreover, investors are subject to the financial constraints. Specifically, $\sigma_i \leq \bar{\zeta}$.*

In equilibrium when the constraint does not bind, σ_A can be expressed as (95) from Lemma 3. Other equilibrium objects behave like the ones in Lemma 3.

When the constraint binds, σ_A equals $\bar{\zeta}$, and the equilibrium objects can be obtained to first order in ϵ as follows.

i. Risk exposure for investors of type B

$$\sigma_B(w;\epsilon) = \frac{\sigma - w\bar{\zeta}}{1-w} + O(\epsilon^2). \tag{99}$$

ii. Market price of risk

$$\eta(w;\epsilon) = \gamma(1+\Delta\gamma)\frac{\sigma - w\bar{\zeta}}{1-w} + O(\epsilon^2). \tag{100}$$

iii. The growth rate of capital

$$g(w;\epsilon) = g^* + q^*(\psi-1)\frac{2w^2\bar{\zeta}^2 - w\bar{\zeta}^2 - 2w\bar{\zeta}\sigma + \sigma^2}{(g^*\theta\psi - q^* - r^*\theta\psi)(1-w) + w\gamma\bar{\zeta}\theta\sigma\psi - \gamma\theta\psi\sigma^2}\frac{\gamma}{2}\Delta\gamma + O(\epsilon^2). \tag{101}$$

All other equilibrium objects can be found in Appendix A.

Proof. See Appendix A. ■

The first observation is that in equilibrium, financial constraints will bind for intermediaries when their wealth share w is sufficiently low. This is because the unconstrained risk exposure of intermediaries (95) is higher than that of households, and it increases as w decreases. Second, when the constraint binds, more risk-averse households become marginal investors. The risk exposure of households (99) increases relative to the unconstrained economy, as intermediaries are forced to sell assets. Importantly, the market price of risk increases relative to the unconstrained economy from Lemma 3, due to limited risk sharing among investors. The constraint changes aggregate investment relative to the unconstrained benchmark. With only one sector, there is no scope for reallocation across sectors.

These simplified economies distinguish changes in aggregate investment from changes in sectoral allocation. Lemma 2 isolates the role of capital composition in a representative-investor economy. Lemmas 3 and 4 show how intermediary wealth and constrained risk sharing affect investment when sectoral composition is fixed. In the full model, intermediary wealth and capital composition jointly determine portfolio choices, risk prices, and whether the risk constraint binds. Its incidence across sectoral claims depends on their marginal contributions to portfolio risk. The resulting investment responses change capital composition and hence the risks investors must share in the future.

D Global solutions of alternative economies

D.1 Solutions of one-sector models with heterogeneous investors

This section describes alternative economies with heterogeneous investors and presents their solutions. They differ from the main model in that the investors have access to one risky asset: a claim to a Lucas 'tree' or a production firm. The demographics and preferences of investors are the same as in the main model.

Consumption-Based Model This section describes the consumption-based model with heterogeneous investors and one Lucas 'tree', and its solution.

Consumption is assumed to follow a standard geometric Brownian motion.

$$\frac{dC_t}{C_t} = \mu dt + \sigma dB_t, \tag{102}$$

where μ and σ are constants.

There is one state variable: the wealth share of intermediaries w. Denote the dividend yield of the claim to the Lucas 'tree' as F_t. The return on the risky asset can be written as

$$dR_t \equiv \frac{d(D/F)}{D/F} + F dt = \mu_R dt + \sigma_R dB_t, \tag{103}$$

where μ_R and σ_R denote the expected return and volatility of the risky asset, respectively. Applying Itô's lemma, we can express μ_R and σ_R as follows:

$$\mu_R = \mu + F - \mu_F - \sigma\sigma_F + \sigma_F^2, \tag{104}$$

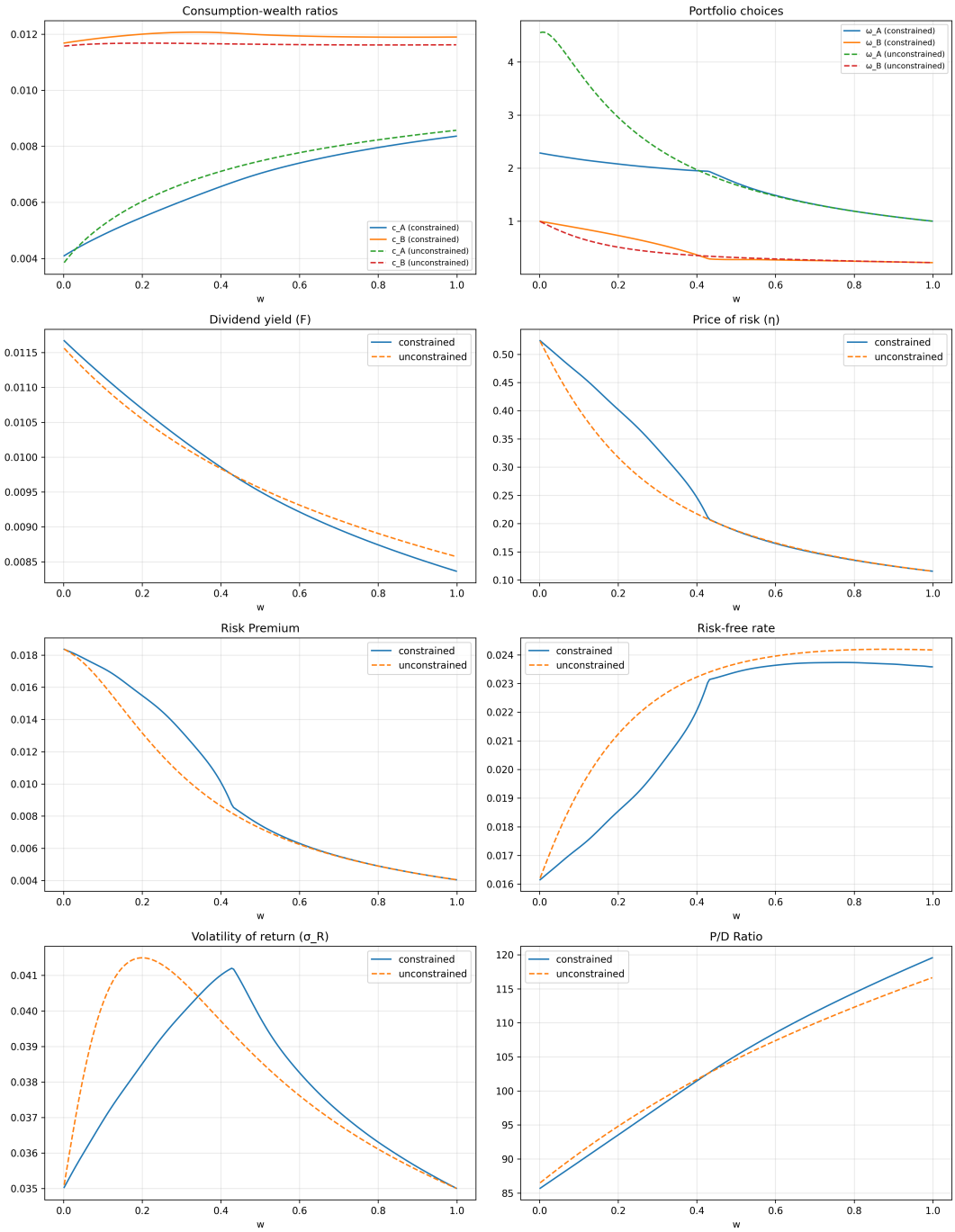
$$\sigma_R = \sigma - \sigma_F. \tag{105}$$

The equilibrium is characterized by two HJB equations for the two types of investors, the PDE that F_t satisfies, and the market clearing conditions. The numerical solution is similar to that in the main model, except that there is only one state variable w. The calibration follows Kargar (2021) closely.

Figure D.1 shows the equilibrium results when the financial constraint is not binding for intermediaries, together with the results from the unconstrained economy for comparison. The solid lines represent the equilibrium with financial constraints, while the dashed lines represent the frictionless equilibrium. The panels in the first row display the consumption-wealth ratios $c_{i,t}$ and portfolio choices $\omega_{i,t}$ for investors of type i. The remaining panels show the dividend yield F_t, price of risk η_t, risk premium $\lambda_{R,t}$, risk-free rate r_t, return volatility $\sigma_{R,t}$,

and price-dividend ratio $1/F_t$. Each quantity is plotted against w_t, the wealth share of intermediaries, which ranges from 0 to 1.

Figure D.1: Solutions of the consumption-based model



This figure shows the consumption-based model solution using the calibrated parameters from Kargar (2021). The solid lines represent the equilibrium with financial constraints, while the dashed lines represent the unconstrained equilibrium. The panels in the first row display the consumption-wealth ratios and portfolio choices for both types of investors. The remaining panels show the dividend yield, price of risk, risk premium, risk-free rate, return volatility, and price-dividend ratio. Each quantity is plotted against w_t, the wealth share of intermediaries.

The frictionless economy is similar to that in Gârleanu and Panageas (2015).

The economy with financial constraints is similar to that in Kargar (2021), but with representative financial intermediaries.

Production-Based Model This section describes the production-based model with heterogeneous investors and one representative firm, and its solution.

The representative firm produces output using capital and the same technology as the firms in the main model. The firm's capital stock evolves according to

$$\frac{dK_t}{K_t} = g_t dt + \sigma dB_t, \tag{106}$$

where σ is a constant, and g_t denotes the expected growth rate of capital at time t.

The equilibrium is characterized by three HJB equations for the firm and two types of investors, along with the market clearing conditions. The numerical solution is similar to that in the main model, except that there is only one state variable w. The calibration follows Silva (2020) closely.

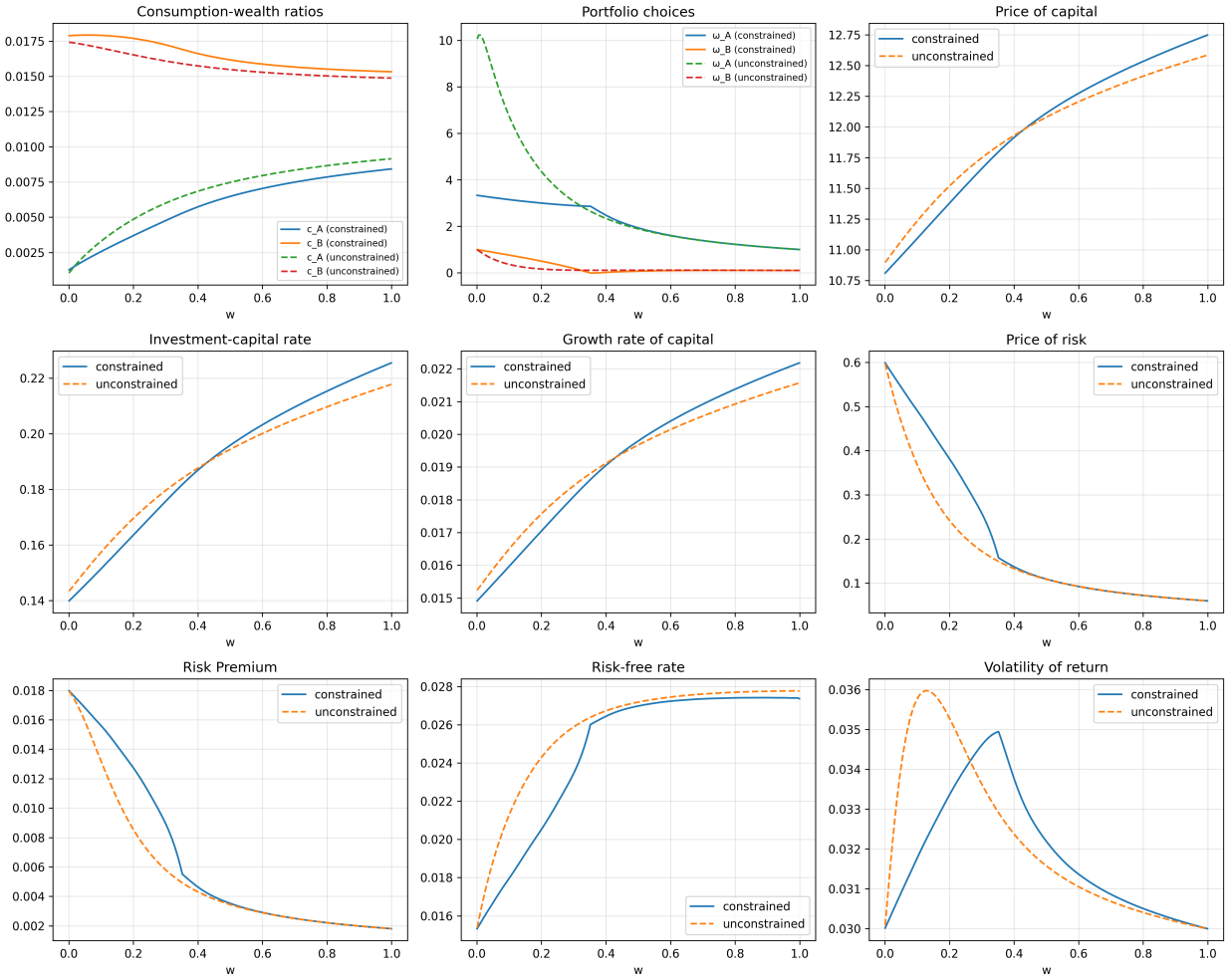
Figure D.2 shows the equilibrium results when the financial constraint is not binding for intermediaries, together with the results from the unconstrained economy for comparison.

D.2 Two-Sector Model

This section describes the economy from Eberly and Wang (2025) that features two sectors, and its solution. It differs from the main model in that there is a representative investor who has access to two risky assets: claims to dividends from firms in two sectors. The investor has the same preferences as in Eberly and Wang (2025).

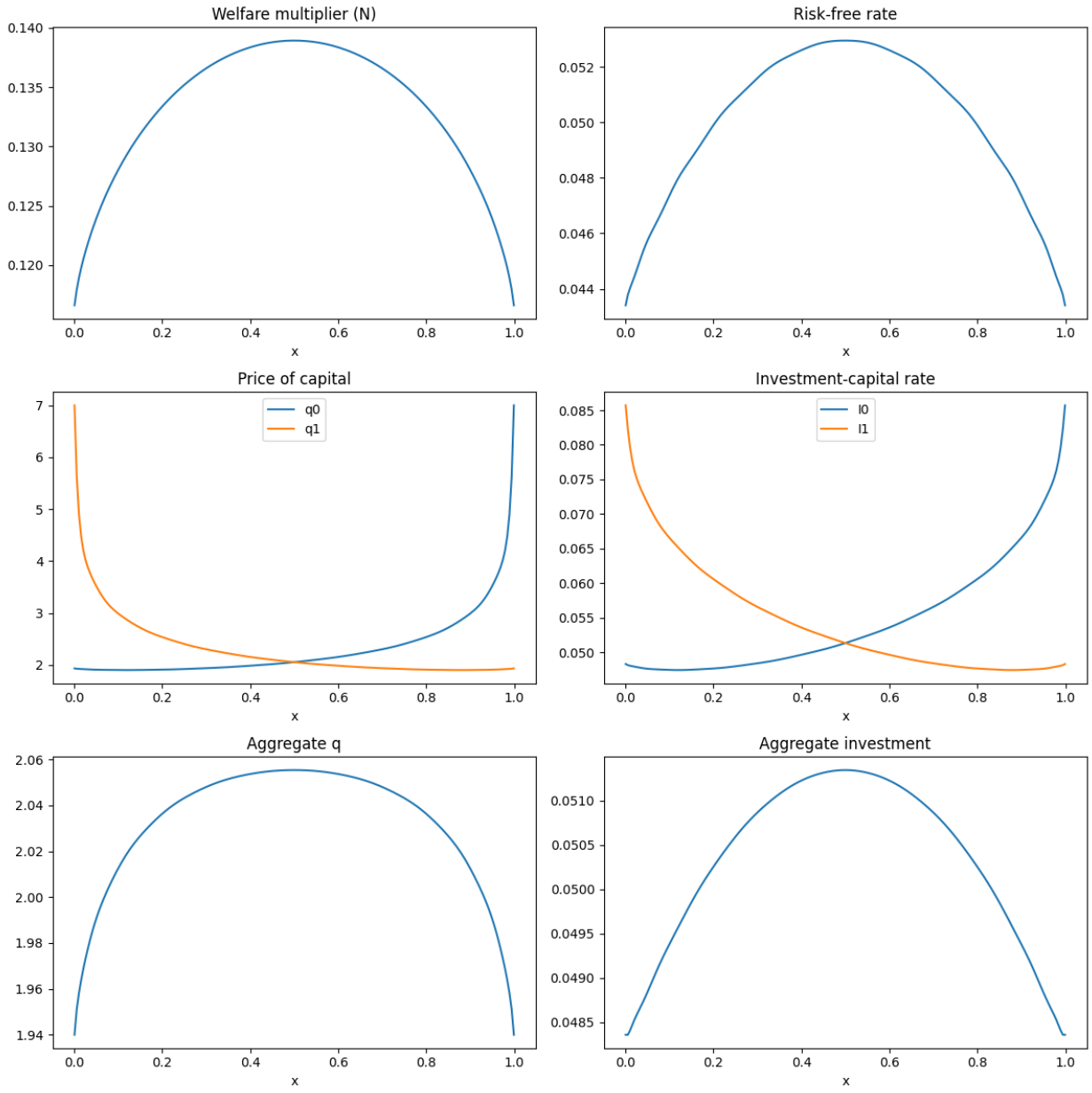
Due to the presence of two sectors and one representative investor, the model has one state variable: the capital share of sector 1, denoted by x. Here I follow Eberly and Wang (2025) and solve the model by invoking the welfare theorem. The social planner's problem can be characterized by a single HJB equation, and N denotes the welfare multiplier. Figure D.3 shows the equilibrium results when the two sectors are symmetric. The panels in the first row display the welfare multiplier of the representative investor and the equilibrium risk-free rate r_t. The second row shows the prices of capital q and investment-capital ratios for the two sectors. The bottom row shows the aggregate price of capital and investment-capital ratio. Each quantity is plotted against x_t, the capital share of sector 1, which ranges from 0 to 1.

Figure D.2: Solutions of the production-based model



This figure shows the production-based model solution using the calibrated parameters from Silva (2020). The solid lines represent the equilibrium with financial constraints, while the dashed lines represent the unconstrained equilibrium. The top and mid panels in the first row display the consumption-wealth ratios and portfolio choices for both types of investors. The remaining panels show the price of capital, investment rate, growth rate of capital, price of risk, risk premium, risk-free rate, and return volatility. Each quantity is plotted against w_t, the wealth share of intermediaries.

Figure D.3: Solutions of the two-sector model



This figure shows the two-sector model solution using the calibrated parameters from Eberly and Wang (2025). The panels in the first row display the welfare multiplier of the representative investor and the equilibrium risk-free rate r_t. The second row shows the prices of capital q and investment-capital ratios for the two sectors. The bottom row shows the aggregate price of capital and investment-capital ratio. Each quantity is plotted against x_t, the capital share of sector 1.

E Additional results for the unconstrained economy

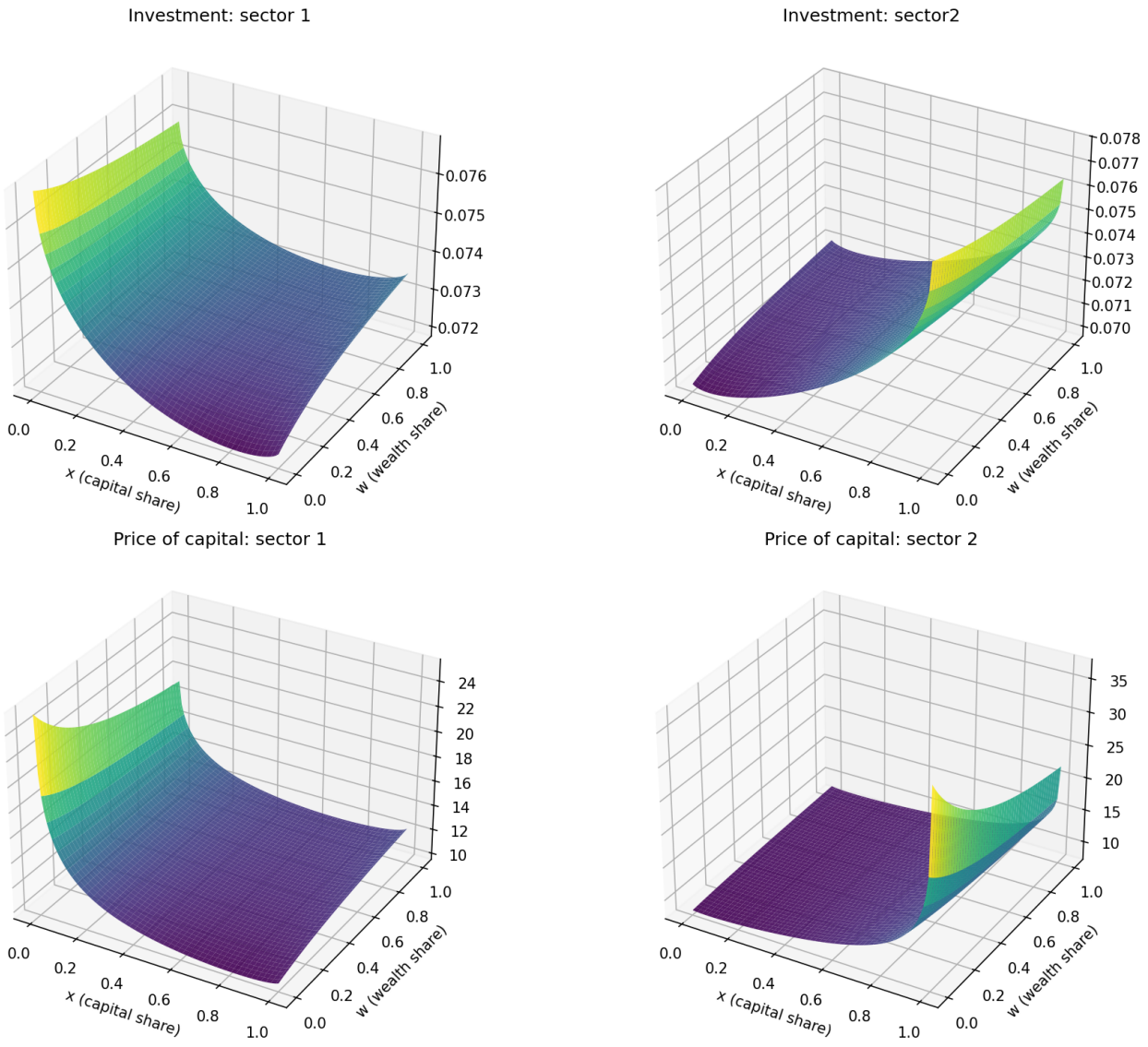
E.1 Sectoral valuations and interest rates

Figure E.1 shows sectoral investment rates and capital prices across the state space. With fixed sectoral productivities, variation in investment reflects the equilibrium valuation of installed capital. When sector 1 is small, its equity can provide diversification even though its capital is more volatile. In some of these states, an increase in intermediary wealth lowers sector 1's valuation. When sector 1 dominates, its valuation generally rises with intermediary wealth. The surface therefore gives a broader view of the role of capital composition than the steady-state slices in Section 5.2.

Figure E.2 shows that the sensitivity of both market prices of risk to capital composition is greater when intermediary wealth share w is low. Changes in the composition of productive risks therefore have larger pricing effects when wealth is held predominantly by more risk-averse households. Proposition 7 connects this interaction to wealth-weighted risk tolerance, aggregate wealth exposures, and investors' hedging demand.

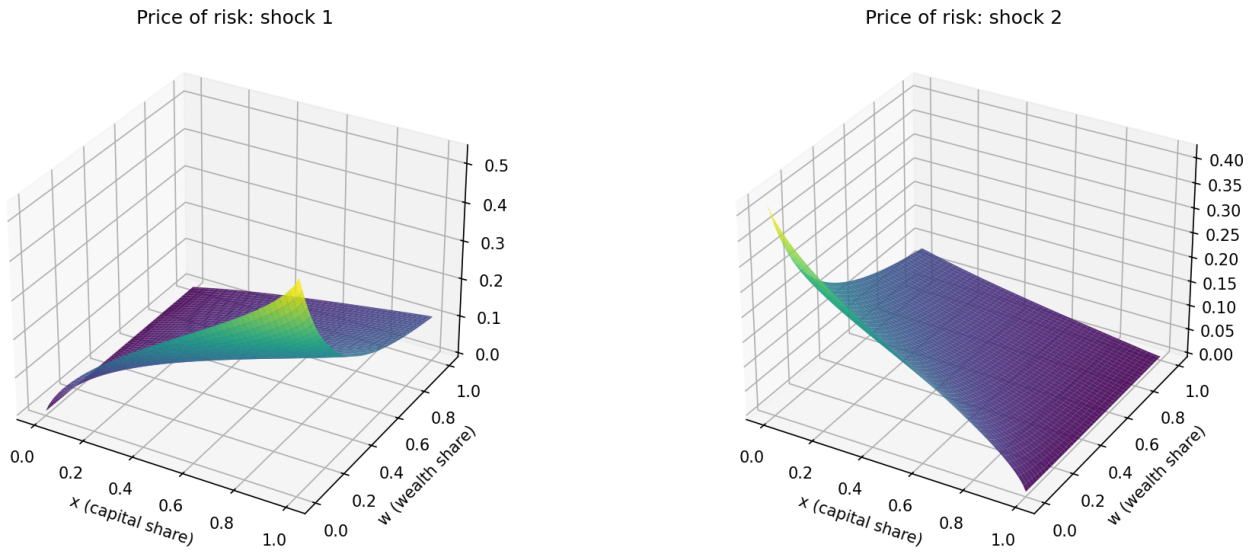
Figure E.3 presents the risk-free rate, which enters the required return on equity alongside risk compensation in (46). The rate generally increases with intermediary wealth share w. Lower rates at low w are consistent with stronger precautionary saving when more risk-averse households hold a larger share of wealth. The rate is also hump-shaped in capital composition x, consistent with weaker precautionary saving when the composition of productive capital provides greater diversification benefits. These interest-rate movements accompany the changes in risk prices and jointly affect sectoral valuations.

Figure E.1: Sectoral dynamics



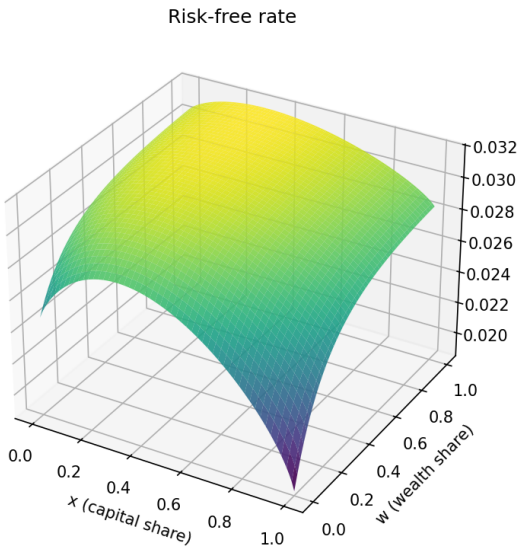
This figure shows the sectoral investment rates and prices of capital (Tobin's Q) as functions of state variables x (capital share of sector 1) and w (wealth share of intermediaries) under the unconstrained calibration. The top panels present the investment rates of the two sectors, and the bottom panels present the prices of capital of the two sectors.

Figure E.2: Prices of risk



This figure shows the prices of risk as functions of state variables x (capital share of sector 1) and w (wealth share of intermediaries) under the unconstrained calibration. The left panel presents the price of risk for shock B_1, and the right panel presents the price of risk for shock B_2.

Figure E.3: Risk-free rate

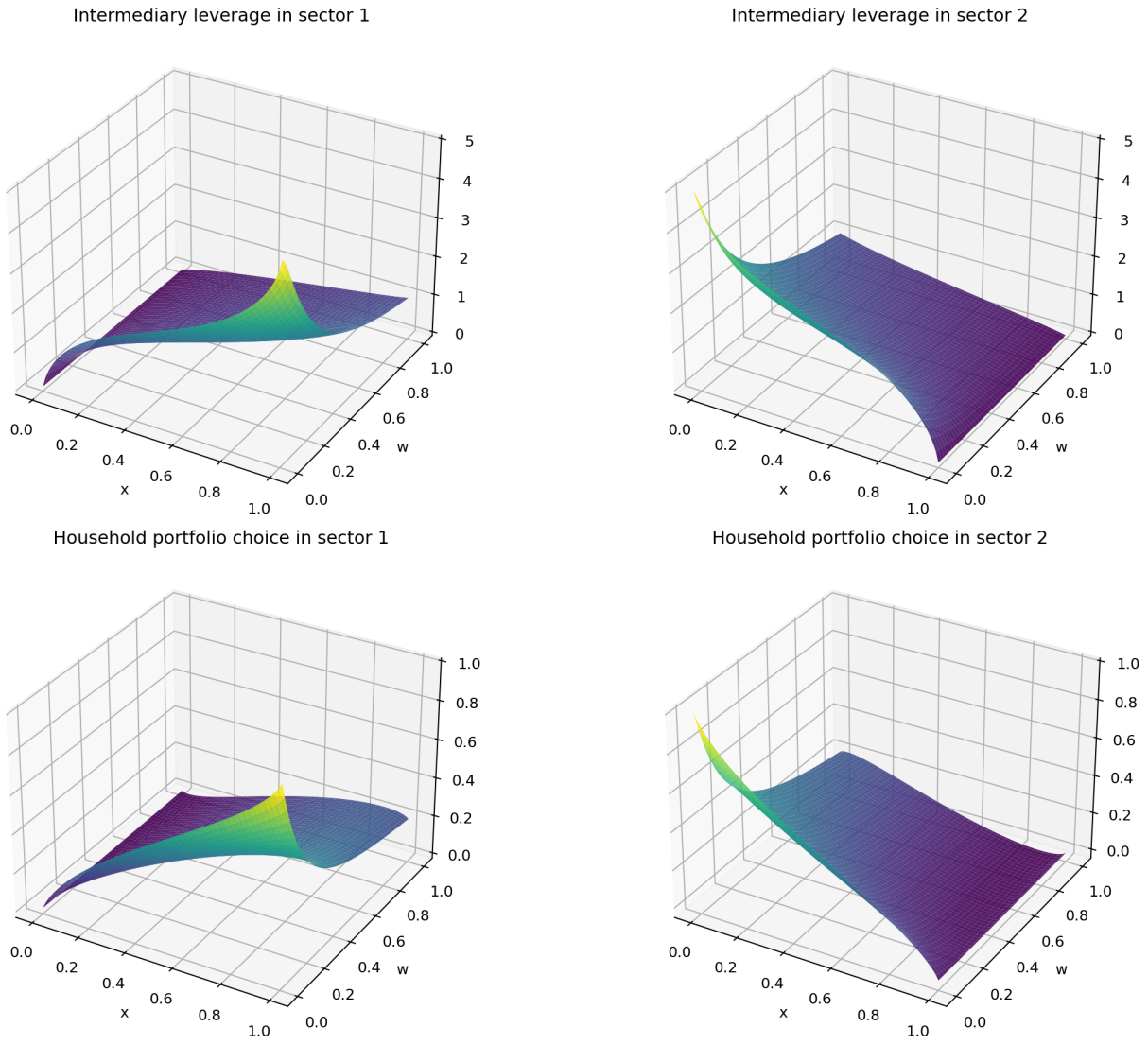


This figure shows the risk-free rate as functions of state variables x (capital share of sector 1) and w (wealth share of intermediaries) under the unconstrained calibration.

E.2 Portfolio holdings and the transmission of sectoral shocks

Figure E.4 reports equity holdings relative to each investor's wealth. Intermediaries hold leveraged portfolios. As sector 1's capital share rises, their portfolio weights shift toward sector 1 and away from sector 2. The risk carried by either position depends on its return loadings on both fundamental shocks.

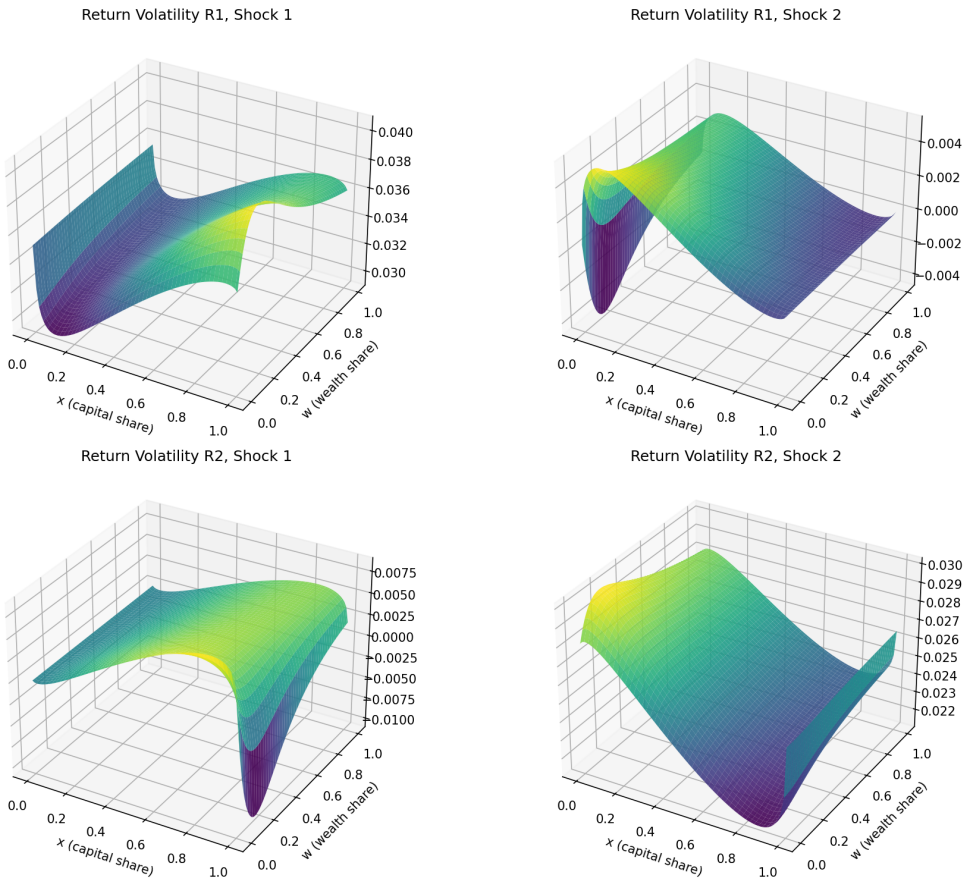
Figure E.4: Portfolio choices



This figure shows sectoral equity holdings relative to investor net worth under the unconstrained calibration as functions of capital composition x and intermediary wealth share w. The top and bottom rows correspond to intermediaries and households, respectively; the left and right columns correspond to sectors 1 and 2. Each panel reports the portfolio weight $\omega_i^{(n)}$, measured as the value of the investor's holdings of sector-n equity divided by the investor's wealth.

Figure E.5 shows the signed loadings of sectoral returns on the two fundamental shocks. Let $\sigma_{R_n}^{(j)}$ denote sector n's return loading on shock j. The bottom-left panel contains regions with a large sector 1 capital share and low intermediary wealth in which $\sigma_{R_2}^{(1)} < 0$. In these states, an adverse sector 1 shock generates a positive instantaneous return on sector 2. This return response incorporates the shock's effects on both x and w, so it identifies a hedge against the shock itself. Where $\sigma_{R_2}^{(1)} > 0$, an adverse sector 1 shock instead generates a loss on sector 2, even though its capital has no direct exposure to that shock. The changing sign of this loading shows how equilibrium valuations can generate either hedging or contagion across sectors.

Figure E.5: Sectoral return exposures



This figure shows the signed return loadings $\sigma_{R_n}^{(j)}$ under the unconstrained calibration as functions of capital composition x and intermediary wealth share w. The top and bottom rows correspond to sectors 1 and 2, respectively; the left and right columns correspond to shocks B_1 and B_2. A negative loading implies a positive instantaneous return response to a negative realization of the corresponding shock.

To connect these return responses to intermediary capitalization, consider the diffusion of wealth share in (14). Denote the coefficient multiplying dB_j in dw by $b_{w,j}$. Using the portfolio

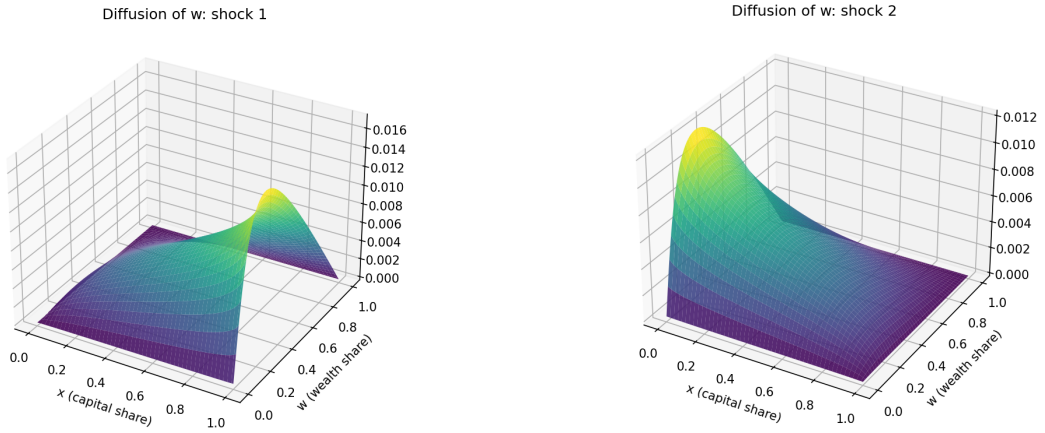
representation of investors' risk exposures gives

$$b_{w,j} \equiv w(1-w)\sigma_w^{(j)} = w(1-w)\sum_{n=1}^{2}\left(\omega_A^{(n)} - \omega_B^{(n)}\right)\sigma_{R_n}^{(j)}, \qquad j \in \{1,2\}. \tag{107}$$

Each asset contributes according to its return loading and the difference between intermediaries' and households' portfolio weights. For example, if $\sigma_{R_2}^{(1)} < 0$ and $\omega_A^{(2)} > \omega_B^{(2)}$, sector 2 makes a negative contribution to $b_{w,1}$, cushioning the fall in intermediary wealth share after an adverse sector 1 shock. A positive contribution reinforces that fall. The portfolio holdings underlying these contributions are shown in Figure E.4.

Figure E.6 shows the resulting net exposures. Both diffusion coefficients are positive in the interior of the state space under the reported calibration: an adverse shock to either sector reduces intermediary wealth share on impact. Intermediary wealth share therefore remains exposed to adverse shocks, despite the hedging opportunities available in some states. At a given wealth share, exposure to the first shock generally increases with x, while exposure to the second decreases. The figure also shows non-monotonicity in w, reflecting both endogenous portfolio and return exposures and the factor $w(1-w)$ in (107).[22] Together, the return-loading and wealth-share figures show how capital composition changes the transmission of shocks through asset prices and intermediary wealth.

Figure E.6: Diffusion of intermediary wealth share



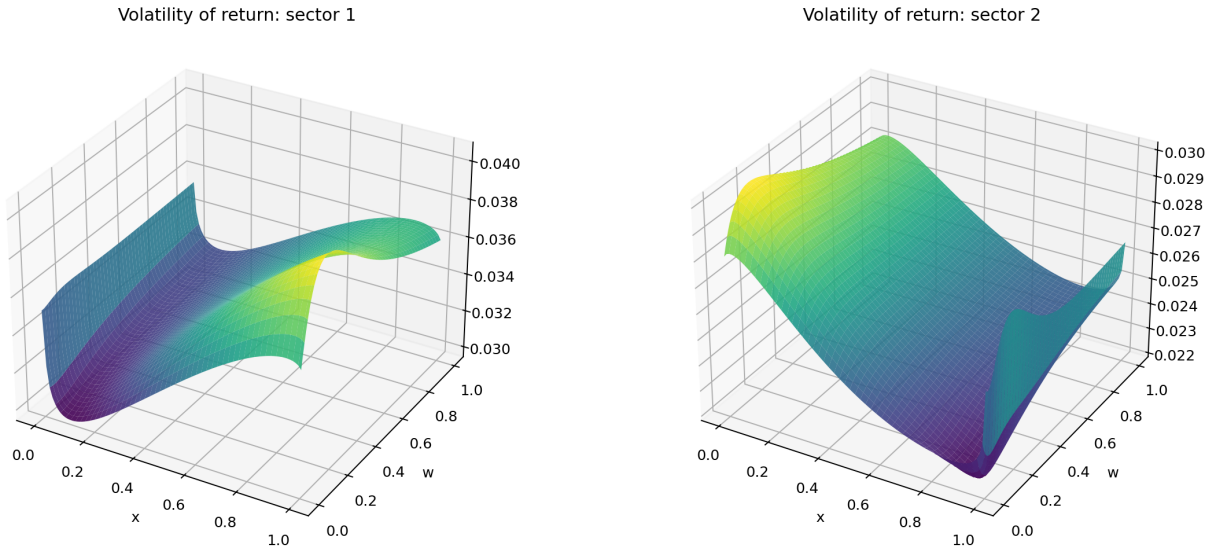
This figure shows the full diffusion coefficients of intermediary wealth share under the unconstrained calibration as functions of x and w. The left and right panels report $b_{w,1} = w(1-w)\sigma_w^{(1)}$ and $b_{w,2} = w(1-w)\sigma_w^{(2)}$, respectively, as defined in (107). These coefficients measure the net instantaneous wealth-share response to each fundamental shock.

Figure E.7 reports total return volatility, calculated as $\|\boldsymbol{\sigma}_{R_n}\|$, the square root of the sum of

[22] Because this factor approaches zero at the wealth-share boundaries, a small diffusion coefficient near a boundary does not by itself indicate weak financial amplification.

squared return loadings. By (26), returns combine the sector's direct fundamental exposure with the response of its valuation to both shocks. Return volatility is non-monotonic in capital composition. For each sector, volatility rises when its capital share becomes very small, where valuations are sensitive to changes in x. At other capital compositions, this valuation response can offset part of the direct fundamental exposure. The resulting volatility patterns reflect the interaction of diversification and the *balance-sheet* channel through equilibrium prices.

Figure E.7: Volatility of returns



This figure shows the volatility of returns as functions of state variables x (capital share of sector 1) and w (wealth share of intermediaries) under the unconstrained calibration. The left panel presents the volatility of returns for sector 1, and the right panel presents the volatility of returns for sector 2.

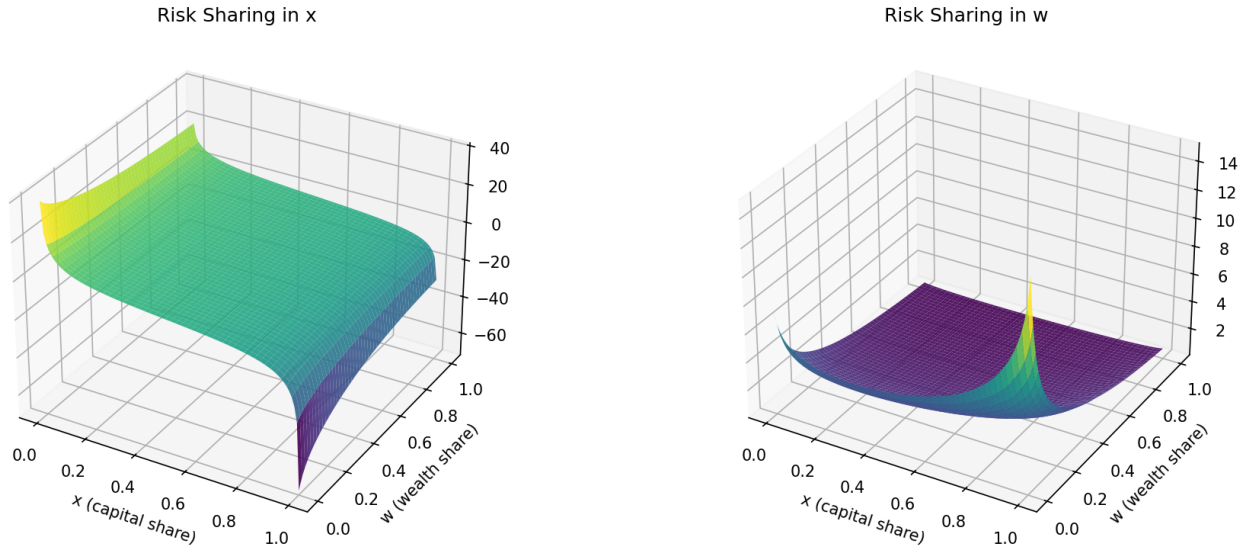
E.3 Hedging demand and portfolio revaluation

Proposition 6 relates portfolio exposures to hedging demand. The portfolio condition (32) combines myopic demand for risk with demand to hedge changes in investment opportunities. Higher market prices of risk at low w raise the myopic component of desired exposure, while hedging demand also depends on how investment opportunities respond to shocks. As (40) shows, R_x and R_w capture differences in how investors' marginal values of wealth respond to changes in capital composition and wealth distribution, respectively. These differences shape investors' hedging demands and thereby influence how aggregate risks are shared.

Figure E.8 shows that R_x is positive when x is close to zero and negative when x is close to one, whereas R_w is positive across the plotted state space. The sign reversal in R_x means that intermediaries' log marginal value of wealth responds more positively to an increase in x than

households' near $x = 0$, and less positively near $x = 1$. Changes in x alter the mix of productive risks and the investment opportunities facing both investors; changes in w redistribute wealth between investors with different risk tolerance. The R_x term therefore captures an additional source of differences in hedging incentives that arises when productive composition can change.

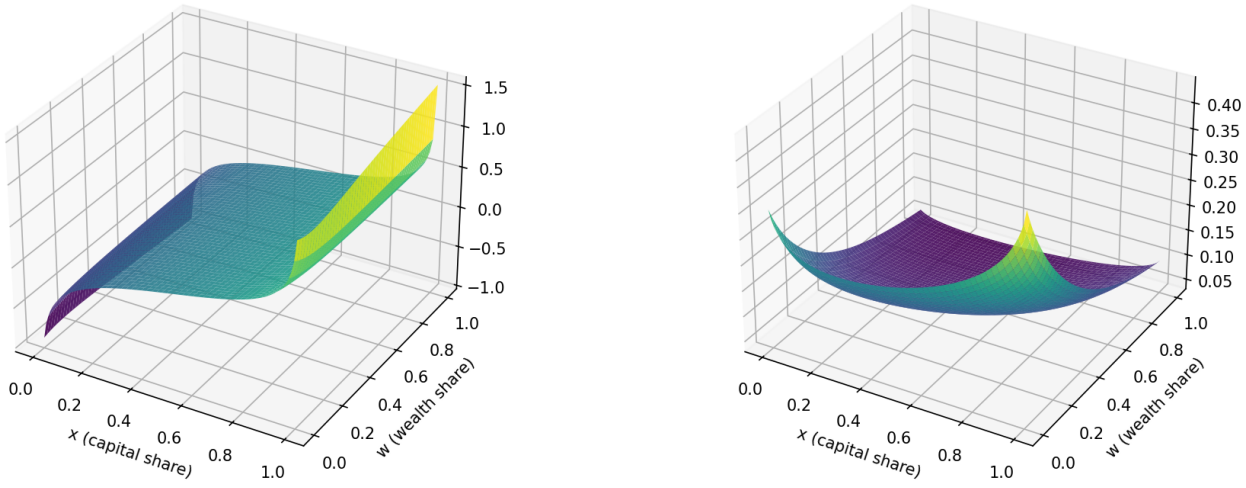
Figure E.8: Risk sharing R_x and R_w



This figure shows the risk-sharing terms from Proposition 6 as functions of state variables x (capital share of sector 1) and w (wealth share of intermediaries) under the unconstrained calibration. The left panel presents the risk-sharing term R_x, and the right panel presents the risk-sharing term R_w.

The portfolio revaluation terms Q_x and Q_w in Proposition 6 describe the valuation side of this interaction. They average the proportional sensitivities of q_1 and q_2 to x and w, respectively, using each sector's share of total equity value as its weight. Through (26), these valuation responses modify the return exposures of the aggregate equity portfolio that investors must hold. Figure E.9 shows that Q_x is negative near $x = 0$ and positive near $x = 1$, with large absolute values near the boundaries. Thus, the valuation component of aggregate risk exposure depends on which sector accounts for most productive capital. The figure also shows that Q_w is positive and generally larger when w is low. Holding capital composition fixed, the positive Q_w implies that weaker intermediary capitalization lowers the market value per unit of aggregate capital, consistent with the *balance-sheet* channel, even though individual sectoral valuations can respond differently.

Figure E.9: Portfolio revaluation Q_x and Q_w



This figure shows the portfolio revaluation terms from Proposition 6 as functions of state variables x (capital share of sector 1) and w (wealth share of intermediaries) under the unconstrained calibration. The left panel presents Q_x, and the right panel presents Q_w.